UNIVERSITÀ DEGLI STUDI DI PAVIA
DIPARTIMENTO DI FISICA
CORSO DI LAUREA MAGISTRALE IN SCIENZE FISICHE

On the complex of observables for linearized gravity

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Anno Accademico 2019/2020

Abstract

La tesi si propone di costruire il complesso delle osservabili per la gravità linearizzata su spazitempi Ricci-piatti. Dapprima si studierà la teoria da un punto di vista classico con l'aiuto di tecniche di teoria dell'omotopia e in seguito se ne svilupperà una quantizzazione nel contesto delle teorie di campo omotope.

The aim of the thesis is to build the complex of observables for linearized gravity on Ricci-flat spacetimes. Initially the theory will be studied in a classical setting with the help of techniques from homotopy theory and then its quantization in the context of homotopy AQFT will be developed.

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Introduction

Gauge theories are of great relevance in physics since they represent the framework used to describe the fundamental interactions. General relativity, which is the state-of-the-art description of gravitation, falls itself into this class of theories. Several attempts at developing a strong axiomatic framework to describe and to quantize them have been made in the past years, see [Dim92; SDH14; BDS14; BDHS14] for electromagnetism and [FH13; BFR13; BDM14] for linearized gravity.

A powerful framework to study quantum field theories on Lorentzian manifolds, *i.e.* on generic curved spacetimes, is Algebraic Quantum Field Theory (AQFT) [HK64; BFV03; BDFY15]. Quantum field theory on curved backgrounds provides a first approximation aimed at combining gravity and quantum theory. As a matter of fact, the gravitational field is treated classically within the framework of general relativity while quantum fields are studied in their propagation on various spacetimes. This leads to difficulties at adapting the usual Hilbert space formalism of quantum mechanics to this scenario. To clarify ideas, a first problem is the impossibility to introduce a unique Hilbert space for a theory which contains all physically relevant states. For example, the Hilbert space which contains the vacuum state for the theory may not contain the thermal state [FR19]. The algebraic formalism solves this problem by dividing the study of a quantum system in two distinct steps. First, an algebra of observables $\mathfrak{A}$ is selected. It contains all information about dynamics, the canonical commutation relations (CCRs) and the causal properties of the system. This is, in a sense, an extension of the Heisenberg picture of quantum mechanics, which emphasizes observables over states. Then, an algebraic state, that is a functional $\rho : \mathfrak{A} \to \mathbb{C}$ which is positive and normalized, needs to be introduced. This allows us to associate to each observable a number, that is interpreted as its expectation value. The state encodes information about all non-local features and correlations of the quantum system. This approach has the advantage of not having to fix an Hilbert space for the system once and for all in order to describe it. Nevertheless, the usual description of quantum mechanics is not lost. In fact, once an algebraic state is fixed the Gelfand-Naimark-Segal theorem [BDH13; HW15] yields an Hilbert space $\mathcal{H}_\rho$, a normalized vector $\psi \in \mathcal{H}_\rho$ and a representation of the algebra $\mathfrak{A}$ over $\mathcal{H}_\rho$ such that the expectation values given by the functional ρ correspond to those calculated according to the Born rule on ψ. Hence, the usual probabilistic interpretation of quantum mechanics is recovered.

The distinguishing feature of gauge theories is that their equations of motion exhibit an invariance property with respect to a suitable class of transformations. This feature is fundamental and it should not be neglected in the study of gauge theories. Hence, they are traditionally quantized in terms of gauge-invariant on-shell observable algebras. This means that the observables do not only implement the dynamics but they also keep track of the gauge symmetry by requiring that they do not distinguish field configurations that are linked by a gauge transformation. However, this approach was observed to be in conflict with some crucial axioms of AQFT. In particular, these violations are linked to the presence of topological charges in quantum gauge theories, *i.e.* electric and magnetic fluxes in Abelian Yang-Mills theory which are expected from a physical point of view [BDS14; BDHS14; Ben16; BBSS16]. Moreover, a second issue, more technical, is related to poor local-to-global features of this formulation of gauge theories. For example, there are some local-to-global constructions, see [DL12; SDH14], that lack to encode features, like Dirac's charge quantization and Aharonov-Bohm phases, that are crucial from a physical point of view. It was later observed that this failure of gauge theories to fulfill the AQFT axioms is due to the categorical structures of both classical and quantum gauge theories which are ignored in the traditional AQFT setting.

An approach meant to resolve this conflict between AQFT and gauge theories has been recently developed. It is the *homotopy AQFT program*, see *e.g.* [BSS15; BSS18; BS19a; BS19b; BBS19; BSW19; BSW20], which suggests to refine the foundations of AQFT in order to implement new notions from category theory. For linear gauge theories, this approach requires field and solution spaces to have the structure of *chain complexes of vector spaces* and quantum observable algebras to have the structure of *differential graded $*$-algebras*. These structures have already been used in the study of gauge theories within the BV-BRST formalism, see [Hol08; FR12a; FR12b] for its review in the algebraic framework. As a matter of fact, chain complexes allow to arrange in a single object both gauge and ghost fields, as well as their antifields. Differential graded $*$-algebras are likewise the appropriate objects to accomodate quantum observables which test gauge fields, ghost fields, and the corresponding antifields as well.

Examples of quantum field theories in the homotopy AQFT approach have been built in [BBS19], where explicit models for Klein-Gordon theory and linear Yang-Mills theory are presented. The aim of this thesis will be to follow this path and to build another working example of a quantum field theory in the homotopical approach. Explicitly, we will consider the gauge theory of linearized gravity on a vacuum spacetime with a vanishing cosmological constant, $\Lambda = 0$. That is the linearization of Einstein's equation.

General relativity plays a fundamental role in physics, providing the description of gravitation. Its quantization is one of the hardest and most debated problems in theoretical and mathematical physics. Many attempts to tackle this problem have been made, but none has yet led to a unanimous solution. The linearization of Einstein's equation is usually seen in combination with the study of classical phenomena, such as gravitational waves [Wal84]. Nevertheless, it was observed that low energy effects of quantum gravity can be studied by means of the quantum field theory of linearized gravity, which considers linear perturbations of the metric as a quantum field propagating on a fixed background spacetime. This approach has found application in cosmology, in the study of fluctuations in the cosmic microwave background, see [Wei08], and models of quantum linearized gravity in a general covariant framework were developed in the past years, see *e.g.* [FH13; BDM14]. Moreover, it was also observed that linearized gravity may represent an important tool for extracting information about the local geometry of the full, non-linear, phase space [BFR13]. Therefore, a thorough and deep analysis of linearized gravity as a quantum field theory may be important also in further developments in quantum gravity.

Our homotopical approach will allow us to deal consistently with all structures deriving from the gauge symmetry of the theory. Linearized gravity has indeed the structure of a gauge theory which is inherited from that of general relativity. In fact, Einstein's equation exhibits a gauge invariance under the action of the group of diffeomorphisms. This symmetry admits an important physical interpretation: It describes the equivalence of all reference frames. As we have already observed, this feature translates to linearized gravity which, therefore, is invariant under the action of the group of linearized diffeomorphisms. This symmetry plays a crucial role in linearized gravity and it should not be neglected. The aim of the homotopical approach is precisely to take into account all information encoded by gauge symmetry and this is accomplished by reconsidering *what* the phase space of a gauge theory is. Explicitly, it is not simply given by the set of all field configurations, but it also needs to contain all arrows given by gauge transformations which link gauge fields. Technically, it is given the structure of a groupoid, *i.e.* a category with all arrows invertible, see [BS19a] for a review.

In this work we follow a homotopical perspective which leads us to write a field complex for linearized gravity. This encodes all information about gauge fields, ghost fields and how the latter act on the former. Afterwards, dynamics, namely linearized Einstein's equation, is imposed through a critical locus construction. This is accomplished by introducing an action functional and by demanding it to be stationary. Dynamics is imposed only in a weak sense, *i.e.* up to homotopy, in compliance with the principles of the homotopical approach and with the mathematical structure of the category of chain complexes. The result of this procedure is the complex of solutions for linearized gravity, see Equation (3.39). A complex of observables which tests the field configurations is then introduced by duality.

The complex of observables comes naturally endowed with a *shifted Poisson structure*. The latter is proved to be trivial in homology thanks to the geometry of globally hyperbolic Ricci-flat manifolds. This will allow us to introduce two classes of homotopies which trivialize the unshifted Poisson structure. They play a role similar to that of retarded/advanced Green operators for the gauge fixed problem. Similar results were also proved to hold true for linear Yang-Mills theory in [BBS19]. Therefore, we will observe that linearized gravity has a very similar behavior to linear Yang-Mills theory and it admits a homotopical treatment analogous to the latter. Several difficulties arise in the analysis of the chain complex of observables and of its homology due to the properties of linearized gravity. As a matter of fact, linearized gravity is sensitive to both topology and geometry of the background manifold unlike linear Yang-Mills theory which sees only the former. In the study of these homologies a distinguished role is played by the

Calabi complex [Kha16; Kha17] which allows us to prove some crucial results about the homologies of the complex of solutions and observables on the subclass of constant curvature background manifolds.

By means of retarded/advanced trivializations we introduce an *unshifted Poisson structure* on the complex of observables. This is crucial for the canonical quantization of linearized gravity. We review the axioms of AQFT, which implement causality and time evolution and we consider their homotopical analogues, see [Yau18; BSW19; BBS19]. These allow us to describe a quantum field theory consistently with all mathematical structures of chain complexes. In order to do this we use techniques from model category theory [Hov07; Hir09] and homotopical category theory [DHKS05]. At the end of this work we build the AQFT functor for linearized gravity on the Ricci-flat spacetime category $\mathsf{Loc}_{\mathrm{Ric}}$, $\mathfrak{A}_{\mathrm{LG}} : \mathsf{Loc}_{\mathrm{Ric}} \to \mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}$, which assigns the dg-algebra of quantum observables to each background. We prove that $\mathfrak{A}_{\mathrm{LG}}$ fulfills the homotopy AQFT axioms and thus it can be interpreted as a quantum field theory.

We also consider the problem of uniqueness, up to weak equivalences, of our quantization prescription. We show that our construction identifies uniquely a quantum field theory on each fixed Ricci-flat spacetime, however a similar result for the theory on the entire category $\mathsf{Loc}_{\mathrm{Ric}}$ is still open. This is due to properties of the category of globally hyperbolic Ricci-flat spacetimes $\mathsf{Loc}_{\mathrm{Ric}}$ as opposed to the slice categories $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$, which should instead be considered when the background spacetime $\overline{M}$ is fixed. In the end, we will also highlight that this homotopical approach yields an AQFT that is not weakly equivalent to other quantizations of linearized gravity that neglect part of the information encoded in our chain complex of observables, see [FH13; BDM14].

An outline of the thesis is the following:

In Chapter 1, a quick overview of the main mathematical notions that are required to develop our formalism will be presented. In particular, we will review the theory of chain complexes, with regards to its model category structure. The main topics will be *chain complexes*, *homology groups* as well as *weak equivalences* and *tensor products*.

In Chapter 2, we will consider vacuum Einstein's equation with vanishing cosmological constant. We will consider the differential equation satisfied by a first order perturbation of the spacetime metric and we will highlight its gauge invariance property. The de Donder gauge will be considered in order to write the equations of motion in a Green hyperbolic form. Afterwards, the space of gauge equivalence classes of on-shell fields will be built. Moreover, gauge invariant on-shell linear observables will also be introduced by a duality argument. Finally, the space of observables will be endowed with a Poisson structure built from the causal propagator for the gauge fixed problem.

In Chapter 3, a different approach to gauge theories will be presented: In order to encompass all information about gauge symmetry, the phase space of the theory will be endowed with the structure of a groupoid. Then, the linear action of gauge transformations will allow us to translate the content of the groupoid for linearized gravity into a chain complex. A critical locus construction will be used to implement dynamics, yielding the complex of solutions of the equation of motion. This complex and its homology groups are interpreted physically in terms of BV-BRST formalism: They contain all gauge and ghost fields, and the corresponding antifields. Finally, the complex of observables for linearized gravity will be introduced by duality. The complex of observables will be studied in depth. In particular, we will observe that it carries naturally a shifted Poisson structure, that can be unshifted by means of retarded/advanced trivializations. Those play a role similar to that played by retarded/advanced Green operators in ordinary field theories.

In Chapter 4, we will deal with the quantization of linearized gravity by enhancing the AQFT formalism to chain complexes. After a short overview of the algebraic formalism, we will build the differential graded algebra of quantum observables for linearized gravity via the canonical commutation relations (CCR). Afterwards, the notion of homotopy AQFT will be introduced. The quantum field theory corresponding to our choice of unshifted Poisson structure will be proved to satisfy the homotopy AQFT axioms on the category of Ricci-flat spacetimes. Furthermore, we will prove that this theory, when restricted to an arbitrary background spacetime, is determined uniquely by our construction.

1 Mathematical tools

1.1 On chain complexes

One of the most recurring structures which appears in the main part of this work is that of chain complexes. We list here the basic notions and results about this topic that are useful later. More in depth analyses and discussions about the theory of chain complexes can be found in [Wei95; Hov07].

Definition 1.1.1 (Chain complex)**.** Let $\mathbb{K}$ be a field of characteristic zero. A *chain complex* is a family of $\mathbb{K}$-vector spaces $(V_n)_{n\in\mathbb{Z}}$ together with a *differential*, *i.e.* a family of linear maps $(\mathrm{d}_n : V_n \to V_{n-1})_{n\in\mathbb{Z}}$ such that $\mathrm{d}_{n-1} \circ \mathrm{d}_n = 0$ for all $n \in \mathbb{Z}$. We will use the following diagrammatic way to represent a chain complex:

$$\cdots \longleftarrow V_{n-1} \xleftarrow{\mathrm{d}_n} V_n \xleftarrow{\mathrm{d}_{n+1}} V_{n+1} \longleftarrow \cdots \tag{1.1}$$

Remark 1.1.2. In order to make notation simpler, we shall denote all data defining a chain complex collectively by V and write $\mathrm{d} : V_n \to V_{n-1}$ for every component of the differential, without explicit reference to the degrees unless ambiguities may arise. ▽

The notion of maps between chain complexes that preserve the chain complex structure is the following one.

Definition 1.1.3 (Chain map)**.** Let V and W be two chain complexes with differentials $\mathrm{d}^V, \mathrm{d}^W$, respectively. A *chain map* $f : V \to W$ is a family of linear maps $(f_n : V_n \to W_n)_{n\in\mathbb{Z}}$ such that $\mathrm{d}^W f_n = f_{n-1}\,\mathrm{d}^V$ for all $n \in \mathbb{Z}$. In other words, the maps are such that the squares in the diagram

$$\begin{array}{ccccccccc}
\cdots \longleftarrow & V_{n-1} & \xleftarrow{\mathrm{d}^V_n} & V_n & \xleftarrow{\mathrm{d}^V_{n+1}} & V_{n+1} & \longleftarrow \cdots \\
& \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} & \\
\cdots \longleftarrow & W_{n-1} & \xleftarrow[\mathrm{d}^W_n]{} & W_n & \xleftarrow[\mathrm{d}^W_{n+1}]{} & W_{n+1} & \longleftarrow \cdots
\end{array} \tag{1.2}$$

are all commutative.

Definition 1.1.4. Let $\mathbb{K}$ be a field of characteristic zero. The *category of chain complexes* $\mathsf{Ch}_{\mathbb{K}}$ is the category whose objects are chain complexes over $\mathbb{K}$ and morphisms are chain maps between them.

To each chain complex $V \in \mathsf{Ch}_{\mathbb{K}}$ one assigns a graded vector space $H_\bullet(V)$ which is its homology.

Definition 1.1.5 (Homology)**.** Let $V \in \mathsf{Ch}_{\mathbb{K}}$ be any chain complex. Its *homology* is the graded vector space $H_\bullet(V) = (H_n(V))_{n\in\mathbb{Z}}$, where

$$H_n(V) \coloneqq \frac{\operatorname{Ker}(\mathrm{d} : V_n \to V_{n-1})}{\operatorname{Im}(\mathrm{d} : V_{n+1} \to V_n)} \,. \tag{1.3}$$

Remark 1.1.6. We will often use the following nomenclature:

- an *n-cycle* is any element in the kernel of differential d_n;
- an *n-boundary* is any element in the image of differential d_{n+1}.

We also introduce the notation: $Z_n \coloneqq \operatorname{Ker} \mathrm{d}_n$ and $B_n \coloneqq \operatorname{Im} \mathrm{d}_{n+1}$. ▽

Remark 1.1.7. Observe that a chain map $f : V \to W$ induces a linear map $H_\bullet(f)$ between homologies,

$$H_n(f) : H_n(V) \longrightarrow H_n(W)\,, \quad [v] \longmapsto H_n(f)[v] \coloneqq [f_n(v)]\,, \tag{1.4}$$

where an arbitrary representative v in the homology equivalence class is chosen. The compatibility property of chain maps, $\mathrm{d}^W f_n = f_{n-1}\,\mathrm{d}^V$, guarantees that f_n sends elements of $Z_n(V)$ to elements of $Z_n(W)$ and elements of $B_n(V)$ to elements of $B_n(W)$. This is tantamount to saying that $H_n(f)$ is a well-defined map. Moreover, the association to a chain complex of its homology is functorial,

$$\begin{aligned} H_\bullet : \mathsf{Ch}_{\mathbb{K}} &\longrightarrow \mathsf{Ch}_{\mathbb{K}}\,, \\ V &\longmapsto H_\bullet(V) \\ (f : V \to W) &\longmapsto (H_\bullet(f) : H_\bullet(V) \to H_\bullet(W))\,, \end{aligned} \tag{1.5}$$

where the homologies are seen as chain complexes with trivial differentials. ▽

We now want to compare chain maps. To this aim the notion of chain homotopy is introduced.

Definition 1.1.8 (Chain homotopy)**.** Let $V, W \in \mathsf{Ch}_{\mathbb{K}}$ be two chain complexes and let $f, p \in \mathsf{Ch}_{\mathbb{K}}(V, W)$ be chain maps. A *chain homotopy* between f and p is a family $(h_n : V_n \to W_{n+1})_{n\in\mathbb{Z}}$ of linear maps such that $\mathrm{d}^W h_n + h_{n-1}\,\mathrm{d}^V = f_n - p_n$, for all $n \in \mathbb{Z}$. Two chain maps $f, p : V \to W$ are said to be *chain homotopic* if there exists a chain homotopy h between them.

Remark 1.1.9. Observe that chain homotopic maps induce the same map at the homology level. Indeed, let $f, p : V \to W$ be chain maps with h a chain homotopy between them. Then, for all $[v] \in H_n(V)$, it holds

$$H_n(\mathrm{d}^W h + h\,\mathrm{d}^V)[v] = [\mathrm{d}^W(h_n v) + h_{n-1}\,\mathrm{d}^V v] = [0]\,, \tag{1.6}$$

since v is an element of $Z_n(V)$ and $\mathrm{d}_n^W(h_n v)$ is an element of $B_{n-1}(W)$. It follows that

$$H_\bullet(f) - H_\bullet(p) = H_\bullet(f-p) = H_\bullet(\mathrm{d}^W h + h\,\mathrm{d}^V) \equiv 0\,, \tag{1.7}$$

hence, the maps $H_\bullet(f) = H_\bullet(p)$ coincide. ▽

The definition of chain homotopy can be rephrased in a more convenient way. Let us start by defining the mapping complex between two chain complexes.

Definition 1.1.10. Let $V, W \in \mathsf{Ch}_\mathbb{K}$ be two chain complexes. The *mapping complex* $\underline{\mathrm{hom}}\,(V,W) \in \mathsf{Ch}_\mathbb{K}$ between them is

$$\underline{\mathrm{hom}}\,(V,W)_n := \prod_{m\in\mathbb{Z}} \mathsf{Lin}(V_m, W_{n+m})\,, \tag{1.8}$$

for all $n \in \mathbb{Z}$, where Lin denotes the vector space of linear maps between vector spaces. The differential of the $\underline{\mathrm{hom}}\,(V,W)$ complex is denoted by ∂ and it is defined as

$$\begin{aligned} \partial : \underline{\mathrm{hom}}\,(V,W)_n &\longrightarrow \underline{\mathrm{hom}}\,(V,W)_{n-1} \\ L &\longmapsto \partial L := \big(\mathrm{d}^W L_m - (-1)^n L_{m-1}\,\mathrm{d}^V\big)_{m\in\mathbb{Z}} \end{aligned} \tag{1.9}$$

for all $L = (L_m)_{m\in\mathbb{Z}} \in \underline{\mathrm{hom}}\,(V,W)_n$, $n \in \mathbb{Z}$.

Observe that a chain map $f : V \to W$ is exactly a 0-cycle in $\underline{\mathrm{hom}}\,(V,W)$, *i.e.* an element $f \in \underline{\mathrm{hom}}\,(V,W)_0$ which satisfies $\partial f = 0$. A chain homotopy h between chain maps f and p is a 1-chain, *i.e.* an element in $\underline{\mathrm{hom}}\,(V,W)_1$, such that $\partial h = f - p$. This clarifies that two chain maps $f, p \in \underline{\mathrm{hom}}\,(V,W)_0$ are chain homotopic if and only if they belong to the same homology class, namely $[f] = [p] \in H_0(\underline{\mathrm{hom}}\,(V,W))$. One can extend this picture to higher homotopies too. If h, h' are two chain homotopies between chain maps f and p, then $h - h' \in \underline{\mathrm{hom}}\,(V,W)_1$ is a 1-cycle in $\underline{\mathrm{hom}}\,(V,W)$, *i.e.* $\partial(h - h') = 0$. Furthermore, given two 1-chains $h, h' \in \underline{\mathrm{hom}}\,(V,W)_1$, a higher chain homotopy between them is a 2-chain $\rho \in \underline{\mathrm{hom}}\,(V,W)_2$ such that $\partial\rho = h - h'$. Again, such a higher homotopy exists if and only if $[h] = [h'] \in H_1(\underline{\mathrm{hom}}\,(V,W))$. The same pattern holds for even higher chain homotopies.

In the category of chain complexes $\mathsf{Ch}_\mathbb{K}$ isomorphisms are those chain maps that are linear isomorphisms in each degree. In many situations being isomorphic is a too strong condition for chain complexes and it is to be more useful to regard as "the same" those chain complexes that have isomorphic homologies. This leads to the following definition.

Definition 1.1.11 (Quasi-isomorphism)**.** Let V, W be two chain complexes. A chain map $f : V \to W$ is a *quasi-isomorphism* if the induced map $H_\bullet(f) : H_\bullet(V) \to H_\bullet(W)$ is an isomorphism between the homologies.

Lemma 1.1.12. *Let $V \in \mathsf{Ch}_\mathbb{K}$. Then, $H_n(V) = 0$ for all $n \in \mathbb{Z}$ if and only if $0 \xrightarrow{\sim} V$ is a quasi-ismorphism.*

Proof. Since 0 is the chain complex with trivial degrees and differential, its homologies are all trivial. Let V be a chain complex such that $0 \xrightarrow{\sim} V$ is a quasi-isomorphism, then Definition 1.1.11 implies immediately that all the homologies $H_n(V)$ are trivial. Vice versa, let $V \in \mathsf{Ch}_{\mathbb{K}}$ be a chain complex whose homologies are all trivial. The zero map $0 : 0 \to V$ clearly induces zero maps on the homologies. These are isomorphisms between homologies, which are all trivial. Thence, $0 : 0 \to V$ is a quasi-isomorphism. □

The idea of considering quasi-isomorphic chain complexes as being the same can be formalized further. In fact, it is possible to endow chain complex category $\mathsf{Ch}_{\mathbb{K}}$ with a model structure whose weak equivalences are the quasi-isomorphisms.

Definition 1.1.13. Let $\mathsf{Ch}_{\mathbb{K}}$ be the category of chain complexes. Then, we define the following classes of morphisms:

- *weak equivalences* are quasi-isomorphisms;
- *fibrations* are chain maps which are surjective on each degree;
- *cofibrations* are chain maps which fulfill the left lifting property with respect to acyclic fibrations.

Remark 1.1.14. The choices in Definition 1.1.13 endow the category $\mathsf{Ch}_{\mathbb{K}}$ with the structure of a model category. The proof of this statement is carried out in [Hov07]. Herein, it is shown that it can be obtained as a cofibrantly generated model category, weak equivalences being given by quasi-isomorphisms, generating cofibrations by morphisms $S^{n-1} \to D^n$ and generating trivial cofibrations by morphisms $0 \to D^n$. The objects S^n and D^n are defined below. ▽

Definition 1.1.15. Let V be a $\mathbb{K}$-vector space, define $S^n(V) \in \mathsf{Ch}_{\mathbb{K}}$ by $S^n(V)_n := V$ and $S^n(V)_k := 0$ for all $k \neq n, k \in \mathbb{Z}$. Similarly, define $D^n(V) \in \mathsf{Ch}_{\mathbb{K}}$ by $D^n(V)_k := V$ if $k = n$ or $k = n-1$, and 0 otherwise. The only non-vanishing differential, d_n, in $D^n(V)$ is the identity.

Remark 1.1.16. In chain complexes both products and coproducts are constructed degreewise. Let $V, W \in \mathsf{Ch}_{\mathbb{K}}$, then their product object is

$$V \prod W = (V_n \times W_n)_{n\in\mathbb{Z}} =: V \times W \,, \tag{1.10}$$

and their coproduct is

$$V \coprod W = (V_n \oplus W_n)_{n\in\mathbb{Z}} =: V \oplus W \,. \tag{1.11}$$

In both cases differentials are given by acting with the appropriate differential on each component, namely $\mathrm{d}^{V\times W} : V_n \times W_n \to V_{n-1} \times W_{n-1}$, $\mathrm{d}^{V\times W}(v, w) = \big(\mathrm{d}^V v, \mathrm{d}^W w\big)$, and $\mathrm{d}^{V\oplus W} : V_n \oplus W_n \to V_{n-1} \oplus W_{n-1}$, $\mathrm{d}^{V\oplus W}(v \oplus w) = \mathrm{d}^V v \oplus \mathrm{d}^W w$.

Not only products and coproducts are obtained from a degreewise construction, but the same is also true for each limit and colimit. Since the category $\mathsf{Vec}_{\mathbb{K}}$ of $\mathbb{K}$-vector spaces has all (small) limits and colimits, then also the category $\mathsf{Ch}_{\mathbb{K}}$ is both complete and cocomplete. Thus, $\mathsf{Ch}_{\mathbb{K}}$ is a model category when equipped with the model structure from Definition 1.1.13. ▽

Remark 1.1.17. Observe that the zero chain complex, *i.e.* the one which has 0 as each component, is both initial and terminal in $\mathsf{Ch}_{\mathbb{K}}$. Moreover, any object in $\mathsf{Ch}_{\mathbb{K}}$ is both fibrant and cofibrant. The fact that every object is fibrant is immediate since fibrations are degreewise surjections, see Definition 1.1.13. In order to check that each object is also cofibrant, it can be useful to observe that every $V \in \mathsf{Ch}_{\mathbb{K}}$ can be decomposed as a suitable coproduct of $S^n(\mathbb{K})$ and $D^n(\mathbb{K})$. Then, it is possible to show that both S^n and D^n are cofibrant in the model category $\mathsf{Ch}_{\mathbb{K}}$. One can conclude since cofibrations are preserved by coproducts. ▽

The category $\mathsf{Ch}_{\mathbb{K}}$ carries also a symmetric monoidal structure.

Definition 1.1.18. The *tensor product* $V \otimes W \in \mathsf{Ch}_{\mathbb{K}}$ of two chain complexes $V, W \in \mathsf{Ch}_{\mathbb{K}}$ is defined as the chain complex with components

$$(V \otimes W)_n \coloneqq \bigoplus_{m\in\mathbb{Z}} V_m \otimes W_{n-m}\,, \tag{1.12}$$

for all $n \in \mathbb{Z}$, together with the differential given by the graded Leibniz rule

$$\mathrm{d}\,(v \otimes w) \coloneqq \mathrm{d}\,v \otimes w + (-1)^m v \otimes \mathrm{d}\,w\,, \tag{1.13}$$

for all $v \in V_m$ and $w \in W_{n-m}$. In the right-hand side of Equation (1.12) the symbol $\otimes$ denotes the usual tensor product of vector spaces.

The tensor product of chain complexes is symmetric via the chain isomorphism

$$\gamma : V \otimes W \to W \otimes V\,, \quad v \otimes w \mapsto \gamma(v \otimes w) \coloneqq (-1)^{|m||k|} w \otimes v\,, \tag{1.14}$$

for all $v \in V_m$ and $w \in W_k$. The chain isomorphism γ is referred to as the *symmetric braiding* in the category $\mathsf{Ch}_{\mathbb{K}}$. The model category of chain complexes endowed with the tensor product, symmetric braiding and mapping complex from Definition 1.1.10 is a closed symmetric monoidal model category, see *e.g.* [Hov07].

Lemma 1.1.19. *Let $\mathsf{Ch}_{\mathbb{K}}$ be the closed symmetric monoidal model category of chain complexes. Then the mapping complex functor $\underline{\mathrm{hom}} : {\mathsf{Ch}_{\mathbb{K}}}^{\mathrm{op}} \times \mathsf{Ch}_{\mathbb{K}} \to \mathsf{Ch}_{\mathbb{K}}$ preserves weak equivalences in both the entries.*

Proof. Since $\mathsf{Ch}_{\mathbb{K}}$ is a closed monoidal model category the monoidal structure $\otimes : \mathsf{Ch}_{\mathbb{K}} \times \mathsf{Ch}_{\mathbb{K}} \to \mathsf{Ch}_{\mathbb{K}}$ is a Quillen bifunctor. For each $V \in \mathsf{Ch}_{\mathbb{K}}$, $V \otimes - : \mathsf{Ch}_{\mathbb{K}} \to \mathsf{Ch}_{\mathbb{K}}$ is a left Quillen functor with right adjoint $\underline{\mathrm{hom}}\,(V, -)$. This follows from the fact that V is cofibrant, see [Hov07, Remark 4.2.3]. From the definition of right Quillen functors and Ken Brown's lemma it follows that $\underline{\mathrm{hom}}\,(V, -)$ preserves weak equivalences between fibrant objects. Since all objects in $\mathsf{Ch}_{\mathbb{K}}$ are fibrant, the functor $\underline{\mathrm{hom}}\,(V, -)$ preserves all weak equivalences. The dual argument implies that weak equivalences are preserved also in the first entry. □

To conclude the section, we fix our sign conventions for shiftings of chain complexes.

Definition 1.1.20. Let $V \in \mathsf{Ch}_{\mathbb{K}}$ be any chain complex and let $p \in \mathbb{Z}$. We define the *p-shift of V* as the chain complex $V[p] \in \mathsf{Ch}_{\mathbb{K}}$ with components $V[p]_n := V_{n-p}$, for all $n \in \mathbb{Z}$, and differential $\mathrm{d}_n^{V[p]} := (-1)^p \, \mathrm{d}_{n-p}^V$.

Remark 1.1.21. From Definition 1.1.20 it follows immediately that $V[p][q] = V[p+q]$ and $V[0] = V$ for all $V \in \mathsf{Ch}_{\mathbb{K}}$ and for all $p, q \in \mathbb{Z}$. $\triangledown$

2 Linearized gravity

In this chapter we study the linearization of Einstein's field equation from a classical point of view. We shall focus mainly on the gauge invariance of the theory and we shall exploit it to construct a suitable algebra of observables for the solutions of the field equations.

In this chapter we follow very closely the analysis in [BDM14] and we present here the main results for the sake of completeness.

2.1 The linearized equation

The starting point of our study is of course Einstein's field equation since the object of our attention will be its linearized version. Because of the centrality of this topic in the development of our work we shall spend some time outlining its main features.

Let us start with the setting of our construction and explain what we mean by *spacetime*.

Definition 2.1.1 (Spacetime)**.** A *spacetime* is a quadruple $(M, g, \mathfrak{o}, \mathfrak{t})$, where (M, g) is a four dimensional Lorentzian manifold, $\mathfrak{o}$ is a choice of orientation on M and $\mathfrak{t}$ is a choice of time-orientation.

In the following, in place of the whole quadruple, we shall refer to a spacetime simply with M, or (M, g) if it will be necessary to make the metric explicit. Moreover, as far as the Lorentzian metric g is concerned, we adopt the signature convention $(-, +, +, +)$. It is worth explaining this definition: It is a well established fact in General Relativity that a spacetime is described by a smooth manifold, and in particular by a Lorentzian one. The Lorentzian metric g equips the manifold with a causal structure. Let $x \in M$ be a generic point on the manifold, then we can consider the tangent space T_xM. The metric g allows us to label each tangent vector $v \in T_xM$ according to the value of $g(v, v)$. In particular, we say that the tangent vector v is *timelike* if $g(v, v) < 0$, *lightlike* if $g(v, v) = 0$ and *spacelike* if $g(v, v) > 0$.

For every point $x \in M$, one can construct a two-folded light cone on T_xM, stemming from the zero vector $0 \in T_xM$, by collecting those tangent vectors that are timelike or lightlike. We have the freedom to identify one of the folds as the collection of *future-directed* vectors. The existence of a global smooth vector field $\mathfrak{t}$ which is timelike at each point allows us to make this choice consistently in a smooth way on M. In other words, we can distinguish in a coherent way "future" and "past".

In order to introduce a causal structure for a Lorentzian manifold we need first to consider the next definition.

Definition 2.1.2. Let (M, g) be a spacetime. A *(piecewise) smooth curve* is a (piecewise) smooth function $\gamma : [0, 1] \to M$. We say that γ is *timelike* (respectively *lightlike*, *spacelike*) if such is the vector tangent to the curve at each point. Moreover, we say that it is *causal* if the tangent vector is nowhere spacelike and it is *future (past) directed* if each tangent vector to the curve is future (past) directed.

Exploiting these structures, we can define on a Lorentzian manifold M the notion of chronological (causal) future and past of a point $x \in M$.

Definition 2.1.3. Let (M, g) be a spacetime and $x \in M$ an arbitrary point. We introduce the *chronological future/past* of x as the set $I_\pm(x)$ of all points $y \in M$ such that there exists a future/past-directed timelike curve $\gamma : [0, 1] \to M$ for which $\gamma(0) = x$ and $\gamma(1) = y$. In the same way, we can introduce the *causal future/past*, denoted by $J_\pm(x)$, by considering causal curves instead of timelike ones. Moreover, for any subset $O \subset M$, its chronological past/future is defined as $I_\pm(O) := \bigcup_{x\in O} I_\pm(x)$. Similarly we define $J_\pm(O)$. Finally, we denote the union of the causal future $J_+(O)$ and of the causal past $J_-(O)$ of O with $J(O)$.

This causal structure is not strong enough to guarantee the absence of pathological situations. As a matter of fact, there are Lorentzian manifolds, such as Anti de Sitter spacetime (AdS), see [HE97], which admits closed timelike or causal curves, which one wants to avoid when concerned with the notion of causality. In order to avoid these problems, it is customary to restrict the attention to a particular class of spacetimes, that are the so-called *globally hyperbolic spacetimes*. We start with introducing two auxiliary notions.

Definition 2.1.4. Let M be a spacetime and $\Sigma \subset M$ be a subset. Then,

i. Σ is called *achronal* if each timelike curve in M intersects Σ at most once;

ii. We call *future/past domain of dependece* $D_\pm(\Sigma)$ the set of points $q \in M$ such that every past/future inextensible causal curve passing through q intersects Σ. We denote with $D(\Sigma) := D_+(\Sigma) \cup D_-(\Sigma)$ the *domain of dependence*.

We give now the definition of globally hyperbolic spacetimes:

Definition 2.1.5 (Globally hyperbolic)**.** A spacetime M is *globally hyperbolic* if there exists a Cauchy surface $\Sigma \subset M$, that is a codimension 1 hypersurface which is an achronal set such that $D(\Sigma) = M$.

It is possible to show that global hyperbolicity implies that the spacetime M admits a foliation where each leaf is diffeomorphic to the Cauchy surface Σ and it is labelled by what can be interpreted as a time coordinate, see [BS05; BGP08] for further details. It is clear that this condition is crucial to give an idea of "initial" conditions and, indeed, it constitutes a sufficient condition for a well-posed Cauchy problem in a lot of concrete situations [BGP08]. The existence and uniqueness of the solutions for a Cauchy problem is typically a desired feature, both from a mathematical and from a physical point of view. We give now a useful definition:

Definition 2.1.6. Let M be a globally hyperbolic spacetime and $F \to M$ be a finite-rank real vector bundle. We call

i. $\Gamma_c(F)$ the space of smooth and *compactly supported* sections of the vector bundle F;

ii. $\Gamma_{sc}(F)$ the space of smooth and *spacelike compact* sections of the vector bundle F, that is $f \in \Gamma_{sc}(F)$ if there exists a compact subset $K \subset M$ such that $\operatorname{supp} f \subset J(K)$;

iii. $\Gamma_{pc/fc}(F)$ the space of smooth and *past/future compact* sections of the vector bundle F, that is $f \in \Gamma_{pc/fc}(F)$ if $\operatorname{supp} f \cap J_{\mp}(x)$ is compact for all $x \in M$;

iv. $\Gamma_{tc}(F) \coloneqq \Gamma_{pc}(F) \cap \Gamma_{fc}(F)$ the space of smooth and *timelike compact* sections of the vector bundle F.

Furthermore, it will be useful to consider the restrictions of linear differential operators to sections with causally restricted supports, as per Definition 2.1.6. Therefore, we introduce the following compact notation:

Definition 2.1.7. Let M be a globally hyperbolic manifold and $P : \Gamma(F) \to \Gamma(F')$ be a linear differential operator between finite-rank vector bundles $F \to M$ and $F' \to M$. We denote by $\operatorname{Ker}_{(-)} P$ the *kernel* of the restriction of P to sections with causally restricted supports, $P : \Gamma_{(-)}(F) \to \Gamma(F')$, where $(-) = c, sc, pc, fc, tc$. Similarly we denote by $\operatorname{Im}_{(-)} P$ the *image* of the restriction of P to sections with causally restricted supports.

We are now ready to write the Einstein's field equation:

$$\operatorname{Ric}(g) - \frac{1}{2} g \operatorname{R}(g) = T \,, \tag{2.1}$$

where $\operatorname{Ric}(g)$ is the Ricci tensor associated with the Levi-Civita connection ∇ for the metric g, $\operatorname{R}(g)$ is the scalar curvature and $T \in \Gamma(\otimes^2_S T^*M)$ is the symmetric stress-energy tensor.

Remark 2.1.8. Here we are assuming a vanishing cosmological constant $\Lambda = 0$, following the perspective in [BDM14]. This choice remarkably simplifies the equations we are dealing with, and we expect that this choice does not play a distinguished role in our reasonings. ▽

Let us simplify our setting and let us assume that we are in a vacuum spacetime. This corresponds to $T \equiv 0$. Therefore, Einstein's equation takes the following form in an arbitrary local chart

$$R_{ab} - \frac{1}{2} g_{ab} g^{cd} R_{cd} = 0 \,. \tag{2.2}$$

It follows immediately that

$$0 = g^{ab} R_{ab} - \frac{1}{2} g^{ab} g_{ab} g^{cd} R_{cd} = - g^{ab} R_{ab} \,, \tag{2.3}$$

that is $\mathrm{R}(g) = 0$. This is tantamount to saying that a Lorentzian metric g that fulfills the vacuum Einstein's equations also fulfills

$$\mathrm{Ric}(g) = 0 \,. \tag{2.4}$$

It follows that, for our purposes, Einstein's equation is exactly given by Equation (2.4). Therefore, we can give the following definition.

Definition 2.1.9 (Physical spacetime)**.** A *physical spacetime* M is a globally hyperbolic spacetime fulfilling equation (2.4).

An important feature of Einstein's equation is its invariance under the action of diffeomorphisms: If M and N are smooth manifolds, $f : M \to N$ is a diffeomorphism and g is a Lorentzian metric on N which solves Einstein's equation, then the pullback $f^* g$ is a Lorentzian metric on M which still fulfills Einstein's equation. In other words, given a physical spacetime (N, g), the pullback along any choice of diffeomorphism $f : M \to N$ yields a spacetime $(M, f^* g)$ which is again a physical spacetime, isometric to the former. This is a direct consequence of the geometric nature of the Ricci tensor, that is $\mathrm{Ric}(f^* g) = f^* \mathrm{Ric}(g)$. From a physical point of view the diffeomorphism invariance of Einstein's equation translates the freedom of choice of the reference frame, which plays a key role in General Relativity. In a slightly more abstract language we can say that Einstein's equation exhibits the structure of a gauge theory where the gauge freedom coincides exactly with the possibility of acting on solutions with arbitrary diffeomorphisms. Therefore, gravity is a gauge theory whose gauge group is the group of diffeomorphisms on spacetimes.

In this work we are not going to study the full Einstein's equation but only its linear counterpart. Thus we need to find a linearization of Equation (2.4) in the following sense. Given a physical spacetime (M, g), we consider a *small* perturbation of the background metric g, namely a one-parameter family of metrics $g_\lambda := g + \lambda h$, $\lambda \in (-\varepsilon, \varepsilon)$, where $\varepsilon > 0$ is an expansion parameter and $h \in \Gamma(\otimes_S^2 T^* M)$. We are interested in the equations fulfilled by h at order $\mathcal{O}(\lambda^2)$.

We start from $\mathrm{Ric}(g_\lambda) = 0$ and we compute choosing a local chart. Exploiting the definition of Ricci tensor and the actual form of the Levi-Civita connection we find

$$R_{ab}(g_\lambda) = \nabla_c^\lambda \Gamma^c_{ba}(g_\lambda) - \nabla_b^\lambda \Gamma^c_{ca}(g_\lambda) = 0 \,, \tag{2.5}$$

where with ∇_a^λ we denote Levi-Civita connection for g_λ and $\Gamma^c_{ab}(g_\lambda)$ are the corresponding Christoffel symbols. To expand these equations in powers of λ we need to go step by step expanding each object in our geometric quantities. We obtain the following identities:

$$g_\lambda^{ab} = g^{ab} - \lambda h^{ab} + \mathcal{O}(\lambda^2)\,, \tag{2.6a}$$

$$\Gamma^c_{ab}(g_\lambda) = \Gamma^c_{ab}(g) + \frac{\lambda}{2} g^{cd}\left(\nabla_a h_{db} + \nabla_b h_{da} - \nabla_d h_{ab}\right) + \mathcal{O}(\lambda^2)\,, \tag{2.6b}$$

where we adopted the convention that all indices are raised with respect to the background metric g (*e.g.* $h^{ab} = g^{ac}g^{bd}h_{cd}$) and we recall that ∇_a is the Levi-Civita connection for g. By recollecting all these identities, a long but straightforward calculation leads to

$$\begin{aligned} R_{ab}(g_\lambda) &= R_{ab}(g) \\ &\quad - \frac{\lambda}{2}\left\{\Box h_{ab} + \mathcal{L}_{(\frac{1}{2}\nabla \operatorname{tr} h - \operatorname{div} h)^\sharp} g_{ab} - R^c{}_a h_{cb} - R^c{}_b h_{ca} + 2R^c{}_{bda} h^d{}_c\right\} + \mathcal{O}(\lambda^2)\,, \end{aligned} \tag{2.7}$$

where $\mathcal{L}_X g$ is the Lie derivative of the metric g along the vector field X, $\cdot^\sharp$ is the musical isomorphism from $\otimes^n T^*M$ to $\otimes^n TM$, $\operatorname{tr} h \coloneqq g^{ab}h_{ab}$ is the trace of h and $(\operatorname{div} h)_a \coloneqq g^{cb}\nabla_c h_{ba}$ is its divergence. It is worth noting that all geometric quantities are computed with respect to the background metric. Thus, the linearized Einstein's equation is obtained restricting our attention to the term at first order in λ in the latter equation. Here we use the hypothesis that the background is Ricci flat to obtain

$$-\frac{1}{2}\left\{\Box h_{ab} + \mathcal{L}_{(\frac{1}{2}\nabla \operatorname{tr} h - \operatorname{div} h)^\sharp} g_{ab} + 2R^c{}_{bda} h^d{}_c\right\} = 0\,. \tag{2.8}$$

We can rewrite Equation (2.8) in a simpler way. We start giving the definitions of some relevant operators.

Definition 2.1.10. Let (M, g) be a physical spacetime. We call

i. *trace* $\operatorname{tr} : \Gamma(\otimes^2_S T^*M) \to C^\infty(M) : h \mapsto \operatorname{tr} h \coloneqq g^{ab}h_{ab}$;

ii. *trace reversal* $I : \Gamma(\otimes^2_S T^*M) \to \Gamma(\otimes^2_S T^*M) : h \mapsto Ih$, such that

$$(Ih)_{ab} \coloneqq h_{ab} - \frac{1}{2} g_{ab} \operatorname{tr} h\,;$$

iii. *Killing operator* $\nabla_S : \Gamma(\otimes^n_S T^*M) \to \Gamma(\otimes^{n+1}_S T^*M) : H \mapsto \nabla_S H$, such that

$$(\nabla_S H)_{i_0 i_1 \dots i_n} \coloneqq \nabla_{(i_0} H_{i_1 \dots i_n)}\,,$$

where we take the normalized symmetrization over the indices in brackets;

iv. *divergence* $\operatorname{div} : \Gamma(\otimes_S^{n+1} T^*M) \to \Gamma(\otimes_S^n T^*M) : H \mapsto \operatorname{div} H$, such that

$$(\operatorname{div} H)_{i_1 \dots i_n} := g^{ab} \nabla_a H_{b i_1 \dots i_n} \,;$$

v. *Riemann operator* $\operatorname{Riem} : \Gamma(\otimes_S^2 T^*M) \to \Gamma(\otimes_S^2 T^*M) : h \mapsto \operatorname{Riem} h$, such that

$$(\operatorname{Riem} h)_{ab} := R_a{}^{cd}{}_b h_{cd} \,.$$

Similar definitions are also given for operators acting on sections of the tangent bundle and of its tensor powers. With a slight abuse of notation we denote both with the same symbols.

Using these operators and recalling the identity $\mathcal{L}_X g_{ab} = \nabla_a X_b + \nabla_b X_a$, $\forall X \in \Gamma(TM)$, going through some manipulations we eventually manage to rewrite Equation (2.8) in the following compact and coordinate-independent form

$$(-\Box + 2 \operatorname{Riem} + 2I\nabla_S \operatorname{div})\, Ih = 0 \,. \tag{2.9}$$

Therefore it will be useful to define the differential operator

$$\begin{aligned} P : \Gamma(\otimes_S^2 T^*M) &\longrightarrow \Gamma(\otimes_S^2 T^*M) \\ h &\longmapsto Ph := (-\Box + 2 \operatorname{Riem} + 2I\nabla_S \operatorname{div})\, Ih \,, \end{aligned} \tag{2.10}$$

in order to simplify the notation.

We eventually come to a precise formulation of our problem: We want to study linearized gravity on a physical spacetime (M, g) as a theory for a dynamical field $h \in \Gamma(\otimes_S^2 T^*M)$ that is a smooth symmetric tensor field of type $(0, 2)$, fulfilling the linearized Einstein's equation (2.9), $Ph = 0$.

The first thing we need to analyze is well-posedness of the problem: We want to investigate if the solution of a Cauchy problem whose initial data are imposed on a Cauchy surface of M exists and is unique. A first indication that this is not the case comes from the principal symbol of the operator P, or rather of $P \circ I$. Let us observe that it is entirely equivalent to study one operator or the other since the following result holds.

Proposition 2.1.11. *The trace reversal I is an involution and consequently an isomorphism.*

Proof. The claim follows from a direct calculation. Let $h \in \Gamma(\otimes_S^2 T^*M)$ be an arbitrary smooth section, then

$$\operatorname{tr} Ih = g^{ab}(Ih)_{ab} = g^{ab}\left(h_{ab} - \frac{1}{2} g_{ab} \operatorname{tr} h\right) = \operatorname{tr} h - 2 \operatorname{tr} h = -\operatorname{tr} h \,. \tag{2.11}$$

Incidentally this reveals where the name *trace reversal* comes from. To show that I is an involution it is now sufficient to compute

$$(I \circ Ih)_{ab} = (Ih)_{ab} - \frac{1}{2} g_{ab} \operatorname{tr} Ih = h_{ab} - \frac{1}{2} g_{ab} \operatorname{tr} h + \frac{1}{2} g_{ab} \operatorname{tr} h = h_{ab} \,. \tag{2.12}$$

It follows immediately that I is an isomorphism since we have explicitly found its (left and right) inverse, which is I itself. $\square$

We recall now the definition of principal symbol of a differential operator.

Definition 2.1.12 (Principal symbol)**.** Let E and F be finite rank vector bundles over a smooth manifold M, and suppose $D : \Gamma(E) \to \Gamma(F)$ is a differential operator of order k whose expression in local coordinates is

$$D = \sum_{|\alpha|\le k} D^\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha} \tag{2.13}$$

where, for each multi-index α and $x \in M$, $D^\alpha(x) : E \to F$ is a vector bundle map. The *principal symbol* of D is the vector map $\sigma_D : T^*M \to \mathrm{Hom}\,(E, F)$ defined, in local coordinates, as follows:

$$\sigma_D(\xi) \coloneqq \sum_{|\alpha|=k} D^\alpha(x)\xi_\alpha\,, \tag{2.14}$$

for all $x \in M$ and $\xi \in T^*_x M$.

The calculation of $\sigma_{P\circ I} : T^*M \otimes \otimes^2_S T^*M \to \otimes^2_S T^*M$ proceeds in the following way: First we work a bit on the explicit form of the differential operator

$$\begin{aligned}(P\circ Ih)_{ab} &= -\Box h_{ab} + 2(I\nabla_S \mathrm{div}\, h)_{ab} + \underbrace{2R_a{}^{cd}{}_b h_{cd}}_{\text{0-order}} \\ &= -\nabla^c\nabla_c h_{ab} + I(\nabla_a\nabla^c h_{cb} + \nabla_b\nabla^c h_{ca}) + \cdots \\ &= -\nabla^c\nabla_c h_{ab} + \nabla_a\nabla^c h_{cb} + \nabla_b\nabla^c h_{ca} - g_{ab}\nabla^d\nabla^c h_{cd} + \cdots\,.\end{aligned}$$

Afterwards we can simplify the expression choosing normal coordinates:

$$\begin{aligned}(P\circ Ih)_{ab} &= -\partial^c\partial_c h_{ab} + \partial_a\partial^c h_{cb} + \partial_b\partial^c h_{ca} - g_{ab}\partial^d\partial^c h_{cd} + \cdots \\ &= (-\delta^c_a\delta^d_b\partial^f\partial_f + \delta^d_b\partial_a\partial^c + \delta^d_a\partial_b\partial^c - g_{ab}\partial^d\partial^c)h_{cd} + \cdots\,.\end{aligned}$$

It follows that

$$\begin{aligned}\sigma_{P\circ I}(\xi)_{ab}{}^{cd} &= -\delta^c_a\delta^d_b\xi^f\xi_f + \delta^d_b\xi_a\xi^c + \delta^d_a\xi_b\xi^c - g_{ab}\xi^d\xi^c \\ &= -g_{ef}\xi^e\xi^f \mathrm{id}_{ab}{}^{cd} + \left(\delta^d_b\xi_a\xi^c + \delta^d_a\xi_b\xi^c - g_{ab}\xi^d\xi^c\right)\,,\end{aligned} \tag{2.15}$$

where id here stands for the identity operator on $\Gamma(\otimes^2_S T^*M)$. According to ordinary results on differential equations on Lorentzian manifolds [BGP08; Bae14] we know that a sufficient condition for the existence and uniqueness of solutions for an initial value problem on a globally hyperbolic Lorentzian manifold, together with existence and uniqueness of the Green operators, is that the problem is written in terms of a normally hyperbolic differential operator. This is an operator whose principal symbol is given by $-g(\xi,\xi)\mathrm{id}$. From the calculations above it is clear that the operator $P\circ I$ is not normally hyperbolic and thus some problems concerning well-posedness of the initial value problem may arise. The solutions of Equation (2.9) cannot be constructed in terms of a Cauchy problem with arbitrary initial data. Let us observe that the term which breaks normal hyperbolicity is the one in brackets in the final expression of the principal symbol. Tracing back its origin we find out that this is due to the $I\nabla_S \operatorname{div}$ term.

We now want to convince you that this issue will not get us into big troubles. Indeed, while we were outlining the principal features of Einstein's equation we highlighted diffeomorphism invariance, which endows General Relativity with the structure of a gauge theory. This is transferred to the level of the linearized theory, which is therefore a gauge theory. To understand how this occurs, let us look at the linearized action of a diffeomorphism. Let $f_\lambda : M \to M$ be a 1-parameter group of diffeomorphisms generated by a field X and let us compute

$$\begin{aligned} f_\lambda^* g &= g + \lambda \frac{\mathrm{d}}{\mathrm{d}\,\tau}\bigg|_{\tau=0} f_\tau^* g + \mathcal{O}(\lambda^2) \\ &= g + \lambda \lim_{\tau\to 0} \frac{f_\tau^* g - g}{\tau} + \mathcal{O}(\lambda^2) \\ &= g + \lambda \mathcal{L}_X g + \mathcal{O}(\lambda^2) = g + 2\lambda \nabla_S X^\flat + \mathcal{O}(\lambda^2)\,, \end{aligned} \tag{2.16}$$

where we have made explicit use of the musical isomorphism $\cdot^\flat : \bigotimes^n TM \to \bigotimes^n T^*M$. Because of invariance under diffeomorphisms, both g and $f_\lambda^* g$ ought to be regarded as being the same at the level of General Relativity. It follows that, as far as the linearized theory is concerned, the gauge freedom is such that two dynamical fields $h, h' \in \Gamma(\otimes_S^2 T^*M)$ are to be considered equivalent if they differ by the action of a linearized diffeomorphism. This means if there exists $\chi \in \Gamma(T^*M)$ such that $h' = h + \nabla_S \chi$. This means that for each solution h of the linearized Einstein's equation (2.9), also $h + \nabla_S \chi$ is a solution for arbitrary $\chi \in \Gamma(T^*M)$. This claim can be proved directly showing that $\nabla_S \Gamma(T^*M) \subset \operatorname{Ker} P$. In order to do this, it is useful to prove the following lemma, which states some identities that will also be very useful in the rest of the work.

Lemma 2.1.13. *Let M be a physical spacetime. Then the following identities hold true:*

i. $\operatorname{tr}(\Box - 2\,\mathrm{Riem}) = \Box \operatorname{tr}$ *on* $\Gamma(\otimes_S^2 T^*M)$*;*

ii. $(\Box - 2\,\mathrm{Riem})I = I(\Box - 2\,\mathrm{Riem})$ *on* $\Gamma(\otimes_S^2 T^*M)$*;*

iii. $(\Box - 2\,\mathrm{Riem})\nabla_S = \nabla_S \Box$ *on* $\Gamma(T^*M)$*;*

iv. $2 \operatorname{div} I \nabla_S = \Box$ *on* $\Gamma(T^*M)$*.*

Proof. The proof is only a matter of direct calculations. Let us start with the first identity, and let $h \in \Gamma(\otimes_S^2 T^*M)$ be an arbitrary section. In a local chart we have

$$\operatorname{tr}(\Box - 2\,\mathrm{Riem})h = g^{ab}(\Box h_{ab} - 2R_a{}^{cd}{}_b h_{cd}) = \Box \operatorname{tr} h + 2R^{cd} h_{cd} = \Box \operatorname{tr} h\,, \tag{2.17}$$

where in the last step we used that g is Ricci-flat. The first claim follows. The second one can be proved along the same lines, using also the identity just shown. For this reason we do not dwell into the details. It is interesting to carry out the proof of the third identity since it uses some structural properties of the covariant derivative. Let $\chi \in \Gamma(T^*M)$:

$$\begin{aligned}((\Box - 2\,\mathrm{Riem})\nabla_S\chi)_{ab} &= \Box(\nabla_S\chi)_{ab} - 2R_a{}^{cd}{}_b(\nabla_S\chi)_{cd} \\ &= \left(\frac{1}{2}\nabla^c\nabla_c\nabla_a\chi_b - R_a{}^{cd}{}_b\nabla_c\chi_d\right) + \left(\frac{1}{2}\nabla^c\nabla_c\nabla_b\chi_a - R_b{}^{cd}{}_a\nabla_c\chi_d\right)\end{aligned} \tag{2.18}$$

Since the second term can be obtained from the first one simply replacing a with b and vice versa, it is sufficient to consider only one of them. Let us compute the following:

$$\begin{aligned}\frac{1}{2}\nabla^c\nabla_c\nabla_a\chi_b &= \frac{1}{2}\nabla^c\left(\nabla_a\nabla_c\chi_b - R^d{}_{bca}\chi_d\right) \\ &= \frac{1}{2}\left(\nabla_a\Box\chi_b - R^d{}_c{}^c{}_a\nabla_d\chi_b - R^d{}_b{}^c{}_a\nabla_c\chi_d - (\nabla^c R^d{}_{bca})\chi_d - R^d{}_{bca}\nabla^c\chi_d\right) \\ &= \frac{1}{2}\left(\nabla_a\Box\chi_b + 2R_a{}^{cd}{}_b\nabla_c\chi_d\right),\end{aligned} \tag{2.19}$$

where in the last step we used that g is Ricci-flat, the symmetries of the Riemann tensor and the second Bianchi identity, which implies $\nabla^c R^d{}_{bca} = 0$. Inserting the last identity in the previous one, we find the statement of the lemma. We conclude by saying that the proof of the fourth identity does not present novelties, hence it is omitted. □

We are now ready to prove the following proposition.

Proposition 2.1.14. *Let M be a physical spacetime. Then $\nabla_S\chi \in \operatorname{Ker} P$, $\forall\chi \in \Gamma(T^*M)$.*

Proof. Using the identities in Lemma 2.1.13, we can calculate

$$\begin{aligned}(-\Box + 2\,\mathrm{Riem} + 2I\nabla_S\,\mathrm{div})\,I\nabla_S\chi &= -I\,(\Box - 2\,\mathrm{Riem})\,\nabla_S\chi + I\nabla_S\Box\chi \\ &= -I\nabla_S\Box\chi + I\nabla_S\Box\chi = 0\,.\end{aligned} \tag{2.20}$$

The statement of the theorem is thus proved. □

Therefore linearized gravity is a gauge theory explaining the lack of hyperbolicity of its dynamical operator P. Keeping in mind what happens in other linear gauge theories, such as electromagnetism, we can hope to turn Equation (2.9) into an hyperbolic one by exploiting the underlying gauge invariance. This is indeed the case and it will be widely explored in the following section.

2.2 Gauge fixed dynamics and Green operators

Let us briefly recall the precise formulation of our problem.

The space of fields is $\Gamma(\otimes^2_S T^*M)$, the vector space of symmetric covariant 2-tensors on a physical spacetime M. The differential operator $P : \Gamma(\otimes^2_S T^*M) \to \Gamma(\otimes^2_S T^*M)$, defined as $P \coloneqq (-\Box + 2\,\mathrm{Riem} + 2I\nabla_S\,\mathrm{div})\,I$, acts on this space. We study the problem given by the equation

$$Ph = 0\,, \quad h \in \Gamma(\otimes_S^2 T^*M)\,. \tag{2.21}$$

We have already shown that this equation admits a gauge symmetry encoded by the following equivalence relation on the space of dynamical variables:

$$h \sim h' \in \Gamma(\otimes_S^2 T^*M) \quad \overset{\text{def}}{\Longleftrightarrow} \quad \exists \chi \in \Gamma(T^*M) : h' - h = \nabla_S \chi\,. \tag{2.22}$$

According to this definition we can construct the off-shell configuration space $\mathcal{C}_{\text{off}}(M)$ of gauge equivalence classes of fields:

$$\mathcal{C}_{\text{off}}(M) \coloneqq \Gamma(\otimes_S^2 T^*M)/\sim\,. \tag{2.23}$$

Remark 2.2.1. Thanks to Proposition 2.1.14 it follows that the operator P descends naturally to an operator on $\mathcal{C}_{\text{off}}(M)$, which, with a slight abuse of notation, we still denote by P. $\triangledown$

Going on with this line of reasoning, we introduce the on-shell configuration space $\mathcal{C}_{\text{on}}(M)$ as the space of gauge equivalence classes which solve Equation(2.21). Explicitly,

$$\mathcal{C}_{\text{on}}(M) \coloneqq \{h \in \Gamma(\otimes_S^2 T^*M) \mid Ph = 0\}/\{\nabla_S \chi \mid \chi \in \Gamma(T^*M)\}\,. \tag{2.24}$$

As we said at the end of the previous section, we can try to turn our problem into an hyperbolic one exploiting a suitable gauge fixing condition. For linearized gravity a very common choice is the so-called de Donder gauge, which plays the same role of the Lorenz gauge in electromagnetism. This consists in the condition $\mathrm{div}\, Ih = 0$. We read directly from the expression for P that this condition reduces the operator to a nicer one, which is hyperbolic (modulo trace reversal). The first thing we have to check is the implementability of this gauge fixing condition, that is, if for each on-shell variable h there exists a gauge equivalent counterpart, which also fulfill the de Donder gauge. This is precisely the content of the following proposition.

Proposition 2.2.2. *For each gauge equivalence class* $[h] \in \mathcal{C}_{on}(M)$ *there exists a representative* $h' \in [h]$ *which solves the problem*

$$P'h' \coloneqq (-\Box + 2\,\mathrm{Riem})\, Ih' = 0\,, \tag{2.25a}$$

$$\mathrm{div}\, Ih' = 0\,. \tag{2.25b}$$

Proof. Let $[h] \in \mathcal{C}_{\text{on}}(M)$ and h be any representative. Suppose that h does not fulfill the gauge fixing condition (2.25b), since, otherwise, there would be nothing to show. Then we look for another representative h', which instead satisfies $\mathrm{div}\, Ih' = 0$. Since every $h' \in [h]$ is of the form $h' = h + \nabla_S \chi$, for some $\chi \in \Gamma(T^*M)$, we have to find a 1-form solving

$$0 = \mathrm{div}\, Ih + \mathrm{div}\, I\nabla_S \chi = \mathrm{div}\, Ih + \frac{1}{2}\Box\chi\,, \tag{2.26}$$

where we used the last identity in Lemma 2.1.13. Such a χ always exists since $\operatorname{div} Ih$ represents a smooth source for the normally hyperbolic operator $\Box$, see [BGP08]. Then $h' := h + \nabla_S\chi$ is in the equivalence class $[h]$ and it satisfies the de Donder condition. Moreover, also Equation (2.25a) is verified since h' fulfills both Equation (2.21) and (2.25b). $\square$

Remark 2.2.3. Observe that the de Donder condition does not fix the full gauge freedom. In fact, there remains the possibility of performing another gauge transformation compatible with the de Donder gauge. In order to show this, and explicitly understand what the residual gauge is, let h, h' be two gauge equivalent configurations both satisfying the de Donder gauge. We can write $h' - h = \nabla_S\chi$ and it holds that $0 = \operatorname{div} I(h'-h) = \operatorname{div} I\nabla_S\chi = \frac{1}{2}\Box\chi$. Therefore the de Donder gauge actually selects an entire family of field configurations which differ only by a gauge transformation $\nabla_S\chi$, where $\chi \in \Gamma(T^*M)$ is such that $\Box\chi = 0$. ∇

We can now make use of the Proposition 2.2.2 to go further in the study of our problem. Every gauge equivalence class of solutions of Equation (2.21) is indeed characterized by means of a solution of the gauge-fixed problem, *cf.* Equation (2.25). The possibility of studying the latter instead of the former represents a great opportunity since the operator $Q := -\Box + 2\,\mathrm{Riem}$, which appears in Equation (2.25a), is normally hyperbolic. It is known, see for example [BGP08; Bae14] for the details, that every normally hyperbolic operator on a globally hyperbolic Lorentzian manifold admits unique retarded and advanced Green operators: $\widetilde{G}_\pm : \Gamma_{tc}(\otimes_S^2 T^*M) \to \Gamma(\otimes_S^2 T^*M)$ such that $\forall \rho \in \Gamma_{tc}(\otimes_S^2 T^*M)$:

i. $Q\widetilde{G}_\pm\rho = \rho$;

ii. $\widetilde{G}_\pm Q\rho = \rho$;

iii. $\operatorname{supp}\left(\widetilde{G}_\pm(\rho)\right) \subseteq J_\pm(\operatorname{supp}(\rho))$.

We recall that $\Gamma_{tc}(\otimes_S^2 T^*M)$ is the space of timelike compact sections of $\otimes_S^2 T^*M$, see Definition 2.1.6, and $J_\pm(K)$ denotes the causal future/past of one subset $K \subseteq M$, as per Definition 2.1.7. Moreover, we can define the so-called *causal propagator* $\widetilde{G} := \widetilde{G}_+ - \widetilde{G}_-$, which allows us to construct the following exact sequence [BGP08; Kha14]:

$$0 \longrightarrow \Gamma_{tc}(\otimes_S^2 T^*M) \xrightarrow{Q} \Gamma_{tc}(\otimes_S^2 T^*M) \xrightarrow{\widetilde{G}} \Gamma(\otimes_S^2 T^*M) \xrightarrow{Q} \Gamma(\otimes_S^2 T^*M) \longrightarrow 0\,. \tag{2.27}$$

Let us introduce the standard pairing between sections of $\otimes_S^n T^*M$:

$$(-,-) : \Gamma(\otimes_S^n T^*M)\check{\times}\Gamma(\otimes_S^n T^*M) \longrightarrow \mathbb{R}\,, \quad (H,\Theta) := \int_M \langle H, \Theta^\sharp\rangle \mu_g\,, \tag{2.28}$$

where $\langle -,-\rangle$ is the dual pairing between $\otimes_S^n T^*M$ and $\otimes_S^n TM$, $\check{\times}$ denotes the subset of the Cartesian product whose pairs have compact overlapping supports and μ_g is the volume form. Once this pairing has been given we can introduce the notion of formal adjoint of a differential operator $T : \Gamma(\otimes_S^n T^*M) \to \Gamma(\otimes_S^m T^*M)$. This is a $T^* : \Gamma(\otimes_S^m T^*M) \to \Gamma(\otimes_S^n T^*M)$, which reads, if existent, $(TH, \Theta) = (H, T^*\Theta)$ for each pair $(H, \Theta) \in \Gamma(\otimes_S^n T^*M) \check{\times} \Gamma(\otimes_S^m T^*M)$. It is possible to show that $I, \Box$ and Riem are formally self-adjoint, namely they coincide with their formal adjoints. This implies that Q is also formally self-adjoint and the same holds true for $P' = (-\Box + 2\,\mathrm{Riem})I$, thanks to Lemma 2.1.13. Finally let us check that ∇_S and $-\operatorname{div}$ are one the adjoint of the other.

Lemma 2.2.4. *For each pair $(h, \chi) \in \Gamma(\otimes_S^2 T^*M) \check{\times} \Gamma(T^*M)$ it holds*

$$(h, \nabla_S \chi) = (-\operatorname{div} h, \chi)\,. \tag{2.29}$$

Proof. Let (h, χ) be a pair as in the statement, then

$$\begin{aligned}(h, \nabla_S \chi) &= \frac{1}{2}\int_M h_{ab} \left(\nabla^a \chi^b + \nabla^b \chi^a\right) \mu_g = -\frac{1}{2}\int_M \left\{(\nabla^a h_{ab})\chi^b + (\nabla^b h_{ab})\chi^a\right\} \mu_g \\ &= -\int_M (\nabla^a h_{ab})\chi^b \mu_g = (-\operatorname{div} h, \chi)\,,\end{aligned} \tag{2.30}$$

where in the second step we used the compactness of the support and in the third one the symmetry of h. □

Since the pairing (2.28) is non degenerate, it turns out to be a powerful tool, as the following calculations reveal. Let $h_1 \in \Gamma_{tc}(\otimes_S^2 T^*M)$ and $h_2 \in \Gamma_c(\otimes_S^2 T^*M)$. It follows

$$\begin{aligned}(I\widetilde{G}_\pm h_1, h_2) &= (I\widetilde{G}_\pm h_1, Qh_2') = (h_1, \widetilde{G}_\mp IQh_2') = (h_1, \widetilde{G}_\mp QIh_2') = (h_1, Ih_2') \\ &= (Ih_1, \widetilde{G}_\mp Qh_2') = (\widetilde{G}_\pm Ih_1, h_2)\,,\end{aligned} \tag{2.31}$$

where we wrote $h_2 = Qh_2'$, thanks to the surjectivity of Q, and where we used that $\widetilde{G}_\pm^* = \widetilde{G}_\mp$. The arbitrariness of the sections implies that $I\widetilde{G}_\pm = \widetilde{G}_\pm I$ on $\Gamma_{tc}(\otimes_S^2 T^*M)$.

We can now use these results to construct Green operators for P', which is the operator we are most interested in since it appears in Equation (2.21). Even though P' is not normally hyperbolic, as a quick computation of its principal symbol can show, the operators $G_\pm := \widetilde{G}_\pm \circ I = I \circ \widetilde{G}_\pm$ satisfy the definition for retarded/advanced Green operators for P'. They share the same support properties of $\widetilde{G}_\pm$ and they verify $G_\pm \circ P'\rho = P' \circ G_\pm \rho = \rho$, for all $\rho \in \Gamma_{tc}(\otimes_S^2 T^*M)$. Since P' is formally self-adjoint, as already observed, the existence of such Green operators is tantamount to saying that P' is Green hyperbolic and therefore its retarded/advanced Green operators are also unique. For the definition of Green hyperbolic operators and their properties see [Bae14]. As for the Green operators of Q, we can define the causal propagator $G := G_+ - G_-$ for P' which yields another exact sequence:

$$0 \longrightarrow \Gamma_{tc}(\otimes_S^2 T^*M) \xrightarrow{P'} \Gamma_{tc}(\otimes_S^2 T^*M) \xrightarrow{G} \Gamma(\otimes_S^2 T^*M) \xrightarrow{P'} \Gamma(\otimes_S^2 T^*M) \longrightarrow 0\,. \tag{2.32}$$

Up to now we have studied the properties of P' and constructed the Green operators $G_\pm$. In other words we have tackled only a part of our problem, namely Equation (2.25a). However, this is not the whole story: In order to characterize the on-shell configurations of the field h, we need to take into account also the constraint given by Equation (2.25b). Following the same argument of [BDM14], which in turn is an adaptation of a strategy developed in [Dim92] for the electromagnetic field, we translate the de Donder gauge fixing condition into a suitable restriction on the space $\Gamma_{tc}(\otimes^2_S T^*M)$ on which the causal propagator G acts. Before proving the main theorem of this section, which gives us a characterization of the space of gauge equivalence classes of solutions of linearized gravity, we need some preliminary results.

Lemma 2.2.5. *Let $G_\pm$ be the retarded/advanced Green operators of P' and $G^\Box_\pm$ the ones for $\Box : \Gamma(T^*M) \to \Gamma(T^*M)$. Then*

$$\operatorname{div} IG_\pm h = -G^\Box_\pm \operatorname{div} h\,, \tag{2.33}$$

for each $h \in \Gamma_{tc}(\otimes^2_S T^*M)$.

Proof. This is another result that follows from the non-degeneracy of the integral pairing as in Equation (2.28). Indeed, let $h \in \Gamma_{tc}(\otimes^2_S T^*M)$ and $\xi \in \Gamma_c(T^*M)$, then

$$\begin{aligned}(\operatorname{div} IG_\pm h, \xi) &= -(IG_\pm h, \nabla_S \xi) = -(h, G_\mp I \nabla_S \xi) = -(h, G_\mp I \nabla_S \Box G^\Box_\mp \xi) \\ &= -(h, G_\mp I(\Box - 2\,\mathrm{Riem})\nabla_S G^\Box_\mp \xi) = (h, G_\mp P' \nabla_S G^\Box_\mp \xi) = (-G^\Box_\pm \operatorname{div} h, \xi)\,,\end{aligned} \tag{2.34}$$

where in the first and the last step we used Lemma 2.2.4, in the third one the identity $\Box G^\Box_\mp = \mathrm{id}$ and in the fourth and fifth some identities of Lemma 2.1.13. The arbitrariness of ξ implies the identity in the statement. □

The following proposition translates the de Donder gauge into restriction on the domain of the causal propagator.

Proposition 2.2.6. *Let $\mathrm{Ker}_{tc}(\operatorname{div}) := \{\varepsilon \in \Gamma_{tc}(\otimes^2_S T^*M) \mid \operatorname{div}\varepsilon = 0\}$. It follows that*

i. for each $\varepsilon \in \mathrm{Ker}_{tc}(\operatorname{div})$, $G\varepsilon$ solves the system (2.25), i.e. *it is a solution of the linearized Einstein's equation in the de Donder gauge;*

ii. for each section $h \in \Gamma(\otimes^2_S T^*M)$ which solves the system (2.25), there exists $\varepsilon \in \mathrm{Ker}_{tc}(\operatorname{div})$ such that $G\varepsilon$ and h differ only by a residual gauge transformation.

Proof. i. Let $\varepsilon \in \mathrm{Ker}_{tc}(\operatorname{div})$. Since G is the causal propagator of P', the Equation (2.25a) is automatically solved by $G\varepsilon$. Furthermore, the equation (2.25b) follows from the Lemma 2.2.5:

$$\operatorname{div} IG\varepsilon = -G^\Box \operatorname{div}\varepsilon = 0\,, \tag{2.35}$$

where $G^\Box := G^\Box_+ - G^\Box_-$ is the causal propagator for $\Box$.

ii. Let $h \in \Gamma(\otimes_S^2 T^*M)$ be a solution of the full problem (2.25), namely $P'h = 0$ and $\operatorname{div} Ih = 0$. The first equation states that $h \in \operatorname{Ker} P' = \operatorname{Im}_{tc} G$, due to exactness of the sequence (2.32). Therefore, there exists $\varepsilon' \in \Gamma_{tc}(\otimes_S^2 T^*M)$ such that $G\varepsilon' = h$. By Lemma 2.2.5, the condition (2.25b) becomes $G^\Box \operatorname{div} \varepsilon' = 0$. Since a counterpart of the exact sequence (2.32) holds also for $G^\Box$ and $\Box$, it follows that $\operatorname{Ker}_{tc} G^\Box = \operatorname{Im}_{tc} \Box$. Hence there exists $\eta \in \Gamma_{tc}(T^*M)$ such that $\operatorname{div} \varepsilon' = \Box\eta$. Let $\varepsilon \coloneqq \varepsilon' - 2I\nabla_S\eta \in \Gamma_{tc}(\otimes_S^2 T^*M)$. First we have to prove that ε belongs to the timelike compact kernel of the divergence:

$$\operatorname{div} \varepsilon = \operatorname{div} \varepsilon' - 2 \operatorname{div} I\nabla_S\eta = \Box\eta - \Box\eta = 0\,, \tag{2.36}$$

where the second step follows from Lemma 2.1.13. Finally, let us show that $G\varepsilon$ and h differ only by a residual gauge transformation: Let us compute

$$G\varepsilon = G\varepsilon' - 2GI\nabla_S\eta = h + 2\nabla_S G^\Box\eta\,, \tag{2.37}$$

where the second step is due to the adjoint of the identity (2.33):

$$(GI\nabla_S\eta, v) = -(\nabla_S\eta, IGv) = (\eta, \operatorname{div} IGv) = -(\eta, G^\Box \operatorname{div} v) = -(\nabla_S G^\Box\eta, v)\,, \tag{2.38}$$

for each $\eta \in \Gamma_{tc}(T^*M)$ and $v \in \Gamma_c(\otimes_S^2 T^*M)$. Thus $G\varepsilon$ and h belong to the same equivalence class, since their difference is of the form $\nabla_S\chi$, with $\chi \coloneqq 2G^\Box\eta$. Furthermore, the gauge transformation that links them is precisely a residual one, since $\Box\chi = 0$.

□

As promised, we can now state a characterization theorem for the space of on-shell gauge equivalence classes.

Theorem 2.2.7. *The map $\phi : \operatorname{Ker}_{tc}(\operatorname{div})/\operatorname{Im}_{tc} P \to \mathcal{C}_{on}(M) : [\varepsilon] \mapsto \phi[\varepsilon] \coloneqq [G\varepsilon]$ is an isomorphism.*

Proof. Our first concern is proving that the statement is meaningful. The quotient $\operatorname{Ker}_{tc}(\operatorname{div})/\operatorname{Im}_{tc} P$ is meaningful since the timelike compact image of P is a subspace of $\operatorname{Ker}_{tc}(\operatorname{div})$. This can be proved observing that, for each $h \in \Gamma(\otimes_S^2 T^*M)$ and $v \in \Gamma(\otimes_S^2 T^*M)$ with compact overlapping supports, it holds that

$$(\operatorname{div} Ph, v) = -(h, P \circ \nabla_S v) = 0\,. \tag{2.39}$$

By invoking the non-degeneracy of the pairing we can draw the sought conclusion. Let us check that the definition of ϕ makes sense. Let $\varepsilon \in \operatorname{Ker}_{tc}(\operatorname{div})$. Since $G\varepsilon$ solves the system (2.25), as stated in Proposition 2.2.6, it identifies a unique equivalence class in $\mathcal{C}_{\text{on}}(M)$. In other words the map $\varepsilon \mapsto [G\varepsilon]$ is well-defined. It remains to show that this map descends to the quotient. We need to show that $\operatorname{Im}_{tc} P$ is mapped to the zero equivalence class. Let $\eta \in \Gamma_{tc}(\otimes_S^2 T^*M)$

$$P\eta \longmapsto [GP\eta] = [G(P' + 2I\nabla_S \operatorname{div} I)\eta] = [-2\nabla_S(G^\Box \operatorname{div} I\eta)] = [0]\,, \tag{2.40}$$

where in the second step we used the adjoint of the identity (2.33). This proves that ϕ is a well-defined linear map.

We show that ϕ is both surjective and injective. Surjectivity is a consequence of Propositions 2.2.2 and 2.2.6. From the first one we know that for each $[h] \in \mathcal{C}_{\mathrm{on}}(M)$ there exists $h' \in [h]$ solving the problem (2.25). Then, there exists $\varepsilon \in \mathrm{Ker}_{tc}(\mathrm{div})$ such that $G\varepsilon$ differs from h' only by a residual gauge transformation. Since ε identifies a unique equivalence class $[\varepsilon] \in \mathrm{Ker}_{tc}(\mathrm{div})/\,\mathrm{Im}_{tc} P$, and $\phi[\varepsilon] = [G\varepsilon] = [h]$, we can draw the conclusion.

We still have to prove that for each $[\varepsilon] \in \mathrm{Ker}_{tc}(\mathrm{div})/\,\mathrm{Im}_{tc} P$ such that $\phi[\varepsilon] = [0] \in \mathcal{C}_{\mathrm{on}}(M)$, it holds $[\varepsilon] = [0]$. Let us choose any representative $\widetilde{\varepsilon} \in [\varepsilon] \in \mathrm{Ker}\,\phi$. Then there exists $\chi \in \Gamma(T^*M)$ such that $G\widetilde{\varepsilon} = \nabla_S \chi$. According to the first statement of Proposition 2.2.6, $G\widetilde{\varepsilon}$ is a solution of linearized gravity in the de Donder gauge. In particular, it holds

$$0 = \mathrm{div}\, IG\widetilde{\varepsilon} = \mathrm{div}\, I\nabla_S\chi = \frac{1}{2}\Box\chi\,, \tag{2.41}$$

where the last step follows from the last identity in Lemma 2.1.13. On account of the exactness of the sequence of $\Box$ and its causal propagator, we find $\alpha \in \Gamma_{tc}(T^*M)$ such that $G^{\Box}\alpha = \chi$. Due to the identity adjoint to Equation (2.33), it holds $G\widetilde{\varepsilon} = \nabla_S G^{\Box}\alpha = G(-I\nabla_S\alpha)$. This means that $\widetilde{\varepsilon} + I\nabla_S\alpha =: \gamma \in \mathrm{Ker}\,G = \mathrm{Im}_{tc} P'$ is of the form $P'\beta$, for a suitable $\beta \in \Gamma_{tc}(\otimes_S^2 T^*M)$. To conclude we use the fact that $\widetilde{\varepsilon} \in \mathrm{Ker}_{tc}(\mathrm{div})$:

$$\begin{aligned} 0 = \mathrm{div}\,\widetilde{\varepsilon} &= -\,\mathrm{div}\, I\nabla_S\alpha + \mathrm{div}\, P'\beta = -\frac{1}{2}\Box\alpha + \mathrm{div}\,(-\Box + 2\,\mathrm{Riem})I\beta \\ &= -\frac{1}{2}\Box\alpha - \Box\,\mathrm{div}\, I\beta = -\Box\left(\frac{1}{2}\alpha + \mathrm{div}\, I\beta\right), \end{aligned} \tag{2.42}$$

where we used the identity $\mathrm{div}\,(-\Box + 2\,\mathrm{Riem})I\beta = -\Box\,\mathrm{div}\, I\beta$, which can be proved along the same line of reasoning followed for the third identity in Lemma 2.1.13. Since the timelike compact kernel of $\Box$ is trivial, it follows $\alpha = -2\,\mathrm{div}\, I\beta$. Hence,

$$\widetilde{\varepsilon} = P'\beta + 2I\nabla_S \mathrm{div}\, I\beta = P\beta\,. \tag{2.43}$$

In other words $\widetilde{\varepsilon} \in \mathrm{Im}_{tc} P$ and then $[\varepsilon] = [0]$. This concludes the proof that ϕ is an isomorphism. $\Box$

This result gives us the following characterization of the classical space of gauge equivalence classes of solutions of the linearized Einstein's equation:

$$\mathcal{C}_{\mathrm{on}}(M) \cong \mathrm{Ker}_{tc}(\mathrm{div})/\,\mathrm{Im}_{tc} P\,. \tag{2.44}$$

The next step is to construct a suitable algebra of gauge invariant on-shell linear observables which can test the solutions of linearized gravity. This construction will be the topic of the next section.

2.3 Classical observables

In this section we shall construct an algebra of observables for linearized gravity. This algebra should implement the gauge symmetry (2.22) and the dynamics (2.21). In other words our goal is to construct a gauge invariant on-shell algebra of observables, dually paired to the space of solutions of linearized gravity.

We follow closely the procedure described in [BDM14]. Let us start with the space of off-shell configurations $\Gamma(\otimes_S^2 T^*M)$. Note that at this level we are not yet considering gauge symmetry. A natural choice for a space dually paired with this one is given by the space of compactly supported smooth sections of the bundle, namely $\Gamma_c(\otimes_S^2 T^*M)$. These spaces are dually paired by means of the integral pairing (2.28). Therefore, every section $\varepsilon \in \Gamma_c(\otimes_S^2 T^*M)$ identifies a linear functional on field configurations through the following definition:

$$\begin{aligned} \mathcal{O}_\varepsilon : \Gamma(\otimes_S^2 T^*M) &\longrightarrow \mathbb{R} \\ h &\longmapsto \mathcal{O}_\varepsilon(h) := (h, \varepsilon)\,. \end{aligned} \tag{2.45}$$

The space of all functionals of this form is a real vector space with the usual operations of pointwise sum and multiplication by scalar. We denote it by $\mathcal{E}^{kin}(M)$. Moreover, this space is isomorphic to the space $\Gamma_c(\otimes_S^2 T^*M)$ itself due to the non-degeneracy of the pairing (2.28). This space of functionals is not yet the correct one because of the following reason: It does not encompass neither the gauge symmetry nor the dynamics, and both of these properties are required to interpret such functionals as proper classical observables for linearized gravity.

We start from the gauge symmetry. The space $\mathcal{E}^{kin}(M)$ contains functionals that distinguish sections $h, h' \in \Gamma(\otimes_S^2 T^*M)$ which are different as sections but belong to the same equivalence class in $\mathcal{C}_{\text{off}}(M)$. This is a behavior we want to avoid since sections in the same gauge equivalence class should be regarded as being the same. This is tantamount to saying that we need to restrict to those functionals that are invariant under gauge transformations. Recalling the definition of gauge equivalence classes (2.22), we can make the gauge-invariance condition explicit: Let $[h] \in \mathcal{C}_{\text{off}}(M)$ be an arbitrary gauge equivalence class and $h', h'' \in [h]$ any two representatives. Then a functional $\mathcal{O}_\varepsilon \in \mathcal{E}^{kin}(M)$ is gauge invariant if and only if $\mathcal{O}_\varepsilon(h') = \mathcal{O}_\varepsilon(h'')$. Since h' and h'' are gauge equivalent, there exists a section $\chi \in \Gamma(T^*M)$ such that $h' - h'' = \nabla_S \chi$ and hence our gauge invariant functional $\mathcal{O}_\varepsilon$ is such that $\nabla_S \chi$ is mapped to zero. We conclude that a functional $\mathcal{O}_\varepsilon \in \mathcal{E}^{kin}(M)$ is gauge invariant if and only if $\mathcal{O}_\varepsilon(\nabla_S \chi) = 0$ for each $\chi \in \Gamma(T^*M)$. The following proposition gives another equivalent characterization for functionals invariant under gauge transformations:

Proposition 2.3.1. *A functional $\mathcal{O}_\varepsilon \in \mathcal{E}^{kin}(M)$ is gauge invariant if and only if* $\operatorname{div} \varepsilon = 0$.

Proof. Let $\mathcal{O}_\varepsilon \in \mathcal{E}^{kin}(M)$. Then $\mathcal{O}_\varepsilon(h) = (h, \varepsilon)$ for all $h \in \Gamma(\otimes_S^2 T^*M)$. The gauge-invariance condition reads that, for all $\chi \in \Gamma(T^*M)$, it holds

$$\mathcal{O}_\varepsilon(\nabla_S \chi) = (\nabla_S \chi, \varepsilon) = (\chi, -\operatorname{div} \varepsilon) = 0\,, \tag{2.46}$$

where, in the second step, we used the fact that ∇_S and $-\operatorname{div}$ are dual to each other, as stated in Lemma 2.2.4. Since the pairing is non-degenerate, we conclude that the above identity is true if and only if $\operatorname{div}\varepsilon = 0$. □

According to Proposition 2.3.1, we identify the space of gauge invariant linear functionals with

$$\mathcal{E}^{inv}(M) = \left\{ \mathcal{O}_\varepsilon \in \mathcal{E}^{kin}(M) \,\middle|\, \operatorname{div}\varepsilon = 0 \right\} \cong \operatorname{Ker}_c(\operatorname{div}) \,. \tag{2.47}$$

The next step consists of implementing the dynamics as in Equation (2.21) at the level of observables. Going deeper into details, our goal is to select a class of functionals in $\mathcal{E}^{inv}(M)$ which is best suited to distinguish between on-shell field configurations. In other words, we are not interested in all those linear functionals that vanish on each solution of $Ph = 0$. Therefore, we declare equivalent two functionals in $\mathcal{E}^{inv}(M)$ if they differ only by a third one vanishing on any section $h \in \Gamma(\otimes^2_S T^*M)$ which solves Equation (2.21). We have now to characterize such equivalence classes of functionals. First, we compute the formal adjoint of the dynamical operator P. Let $h \in \Gamma(\otimes^2_S T^*M)$ and $\varepsilon \in \Gamma(\otimes^2_S T^*M)$ be arbitrary sections with compact overlapping supports:

$$(Ph, \varepsilon) = ((P' + 2I\nabla_S \operatorname{div} I)h, \varepsilon) = (h, P'\varepsilon) + (h, 2I\nabla_S \operatorname{div} I\varepsilon) = (h, P\varepsilon)\,, \tag{2.48}$$

where we used Lemma 2.2.4, the formal self-adjointness of the operator P' and of the trace reversal I. Consider now an observable in $\mathcal{E}^{inv}(M)$ associated with a section $\varepsilon \in \operatorname{Im}_c P$, namely $\varepsilon = P\varepsilon'$ for $\varepsilon' \in \Gamma_c(\otimes^2_S T^*M)$. It follows that

$$\mathcal{O}_\varepsilon(h) = (h, P\varepsilon') = (Ph, \varepsilon') = 0\,, \tag{2.49}$$

for each $h \in \Gamma(\otimes^2_S T^*M)$ such that $Ph = 0$. In order to take into account the dynamics, we have to take the quotient between the gauge invariant observables and the image of P. The space of classical observables is

$$\mathcal{E}(M) := \frac{\mathcal{E}^{inv}(M)}{\operatorname{Im}_c P} = \frac{\operatorname{Ker}_c(\operatorname{div})}{\operatorname{Im}_c P}\,. \tag{2.50}$$

Remark 2.3.2. This quotient is well-defined due to the inclusion $\operatorname{Im}_c P \subset \operatorname{Ker}_c(\operatorname{div})$, which is a direct consequence of the gauge symmetry of the equations. We recall that we have already dealt with the well-posedness of a very similar quotient in the proof of the Theorem 2.2.7. ▽

Remark 2.3.3. An element of $\mathcal{E}(M)$ is an equivalence class of functionals whose action is well-defined only on the (gauge equivalence classes of) solutions of the linearized Einstein's equation. To show this point let us reveal how the evaluation of those observables concretely happens. Every $[\varepsilon] \in \operatorname{Ker}_c(\operatorname{div})/\operatorname{Im}_c P$ identifies an observable $\mathcal{O}_{[\varepsilon]}$ whose action on $\mathcal{C}_{\mathrm{on}}(M)$ is given by selecting two arbitrary representatives, namely:

$$\mathcal{O}_{[\varepsilon]}([h]) = \mathcal{O}_\varepsilon(h) = (h, \varepsilon)\,, \tag{2.51}$$

with $h \in [h]$ and $\varepsilon \in [\varepsilon]$. This definition is well-posed and it does not depend on the choice of representatives just because $\mathcal{O}_\varepsilon$ is gauge invariant and h is such that $Ph = 0$.

Let $\varepsilon' \in [\varepsilon]$ be another representative. Then there exists $\eta \in \Gamma_c(\otimes_S^2 T^*M)$ such that $\varepsilon' = \varepsilon + P\eta$. Therefore

$$\mathcal{O}_{\varepsilon'}(h) = (h, \varepsilon + P\eta) = (h, \varepsilon) + (Ph, \eta) = (h, \varepsilon) = \mathcal{O}_\varepsilon(h)\,. \tag{2.52}$$

It is thus clear that the definition is well posed. ▽

The space of classical observables $\mathcal{E}(M)$ can be endowed with a (constant) Poisson structure, which is built with the integral pairing (2.28) and the causal propagator G associated with the gauge-fixed dynamics operator P'.

Proposition 2.3.4. *The map $\tau : \mathcal{E}(M) \otimes \mathcal{E}(M) \to \mathbb{R}$ defined as*

$$\tau([\varepsilon], [\eta]) \coloneqq 2(\varepsilon, G\eta)\,, \tag{2.53}$$

where arbitrary representatives in the equivalence classes appear in the right-hand side, is a well-defined blinear and skew-asymmetric map that endows the space of observables with a Poisson structure.

Proof. First, we prove that τ is a Poisson structure on $\mathcal{E}^{inv}(M)$. From its actual definition we see immediately that τ is bilinear. Moreover, it is skew-symmetric since

$$(\varepsilon, G\eta) = -(G\varepsilon, \eta) = -(G\varepsilon, \eta) = -(\eta, G\varepsilon)\,, \tag{2.54}$$

where we used $G^* = -G$ and the symmetry of the pairing (2.28). This makes τ a Poisson structure on $\mathcal{E}^{kin}(M)$, hence on its subspace $\mathcal{E}^{inv}(M)$. Finally, we have to show that τ descends to the quotient space $\mathcal{E}(M)$. It is sufficient to prove that our definition does not depend on the choice of representative in the first entry. The same holds true also for the second entry due to the skew-symmetry of τ. Since two observables $\varepsilon, \varepsilon'$ are equivalent in $\mathcal{E}(M)$ if and only if there exists $\zeta \in \Gamma_c(\otimes_S^2 T^*M)$ such that $\varepsilon' = \varepsilon + P\zeta$, the Poisson structure is well-defined on the quotient if $\tau(P\zeta, \eta) = 0$ for each $\zeta \in \Gamma_c(\otimes_S^2 T^*M)$ and $\eta \in \mathrm{Ker}_c\,(\mathrm{div})$. Yet it holds

$$\begin{aligned}(P\zeta, G\eta) &= (P\zeta, G\eta) = (\zeta, PG\eta) = (\zeta, (P' + 2I\nabla_S \mathrm{div}\, I)G\eta) \\ &= (\zeta, 2I\nabla_S \mathrm{div}\, IG\eta) = (\zeta, -2I\nabla_S G^\Box\, \mathrm{div}\, \eta) = 0\,,\end{aligned} \tag{2.55}$$

where we exploit that G is the causal propagator of P', Lemma 2.2.5 and $\mathrm{div}\, \eta = 0$. Therefore, τ is a Poisson structure on $\mathcal{E}(M)$. □

Up to this point, we have studied linearized gravity considering its phase space and characterizing the space of gauge equivalence classes of on-shell field configurations via the study of the gauge-fixed dynamics in the de Donder gauge. Furthermore, we have built a space of on-shell gauge invariant observables dually paired with the configuration space. Finally, we have shown that the space of observables can be endowed with a Poisson structure, built only from the structural properties of the differential operator P', ruling the gauge-fixed dynamics. The Poisson structure on $\mathcal{E}(M)$ will be crucial in the quantization of the theory.

Unfortunately, this approach does not grasp some details about the action of the gauge group that are neglected here. In the next chapter we intend to unveil these features.

3 The homotopical approach

In this chapter we develop a different approach for the study of linearized gravity. Following some recent works, [BSS15; BBS19; BS19a], we adopt an "homotopical approach" in order to properly encode the gauge invariance encoded in linearized gravity. This consists in working with a higher categorical structure that goes beyond the naïve gauge orbit space description.

Here we deal only with the classical theory, adapting to the case in hand the results and reasonings developed in [BBS19] for a linear Yang-Mills theory.

The structure of the chapter will be the following: First, we give to the space of gauge fields a richer structure codified in a chain complex. Subsequently, we impose the dynamics through a derived critical locus construction and we calculate explicitly the solution complex. Finally, a complex of linear observables for linearized gravity is computed and it is endowed with an unshifted Poisson structure.

3.1 A groupoid for linearized gravity

In the previous chapter, we have studied linearized gravity considering the gauge orbit space $\mathcal{C}_{\text{off}}(M)$ as the space of off-shell gauge fields for the theory. This leads however to some problems. In particular, it comes out of a simplification of the full gauge structure of the theory and it fails necessarily to encode some information.

As a matter of fact, in the gauge orbit space we codify equivalence classes. The information on gauge transformations and on their action as a link between the equivalent fields, is lost.

Addressing this issue requires to reconsider the concept of gauge field space.

More concretely, let us consider a matter field, say, a real Klein-Gordon field with mass $m > 0$. The underlying space is rather simple: Configurations lie in $C^\infty(M, \mathbb{R})$ and no other structures are required.

As far as linearized gravity (or any other gauge theory) is concerned, the situation is a little trickier. To give a section of the symmetric, totally covariant, 2-tensor bundle it is not enough to exhaust the structure of the configuration space. Indeed, it is also crucial to say how gauge transformations link equivalent fields. Therefore, we have to attach to the space of sections the maps that implement the gauge symmetry. In this way, the space of fields for linearized gravity does not have simply the structure of a set, rather that of a groupoid.

Definition 3.1.1 (Groupoid)**.** A *groupoid* G is a small category whose morphisms are all isomorphisms.

Explicitly, a gauge field groupoid has gauge fields as objects and gauge transformations as morphisms.

The case of linearized gravity on a physical spacetime M is immediately defined:

$$\mathsf{G}^{LG} := \begin{cases} \mathsf{Obj}: & h \in \Gamma(\otimes_S^2 T^*M) \\ \mathsf{Mor}: & h \xrightarrow{\chi} h + \nabla_S \chi\,, \quad \text{with } \chi \in \Gamma(T^*M) \end{cases} . \tag{3.1}$$

Instead of using groupoids, it is possible to accommodate gauge fields and gauge transformations into another mathematical structure, closely related to groupoids. The structure we are referring to is that of a simplicial set. Let us start with some definitions.

Definition 3.1.2 (Simplex category)**.** The *simplex category* Δ is the category whose objects are non-empty, finite, totally ordered sets and whose morphisms are the weakly order-preserving maps between them.

Remark 3.1.3. It is common to refer to a skeleton of Δ, where a fixed representative in each isomorphism class of objects is selected. In this way, the objects of (a skeleton of) Δ are in bijection with natural numbers $n \in \mathbb{N}$ and one writes

$$[n] = \{0 \to 1 \to \cdots \to n\}\,. \tag{3.2}$$

The set of morphisms $\Delta([n],[k])$ from $[n]$ to $[k]$ consits of the weakly order-preserving maps $f : [n] \to [k]$, so that $x \leq y$ implies $f(x) \leq f(y)$. These morphisms are generated by the following two elementary kinds of maps:

i. *coface maps*: $\partial_n^i : [n-1] \to [n]$, for $n \geq 1$ and $0 \leq i \leq n$, where the image of ∂_n^i does not contain $i \in [n]$;

ii. *codegeneracy maps*: $\varepsilon_n^i : [n+1] \to [n]$, for $n \geq 0$ and $0 \leq i \leq n$, where ε_n^i sends $i, i+1 \in [n+1]$ to $i \in [n]$.

All these maps are subject to the following relations, called *cosimplicial identities*:

$$\begin{aligned} &\partial_{n+1}^j \partial_n^i = \partial_{n+1}^i \partial_n^{j-1}\,, && 0 \leq i < j \leq n+1 \\ &\varepsilon_n^j \varepsilon_{n+1}^i = \varepsilon_n^{i-1} \varepsilon_{n+1}^j\,, && 0 \leq j < i \leq n+1 \\ &\varepsilon_n^j \partial_{n+1}^i = \begin{cases} \partial_n^i \varepsilon_{n-1}^{j-1}\,, & 0 \leq i < j \leq n \\ \mathrm{id}_n\,, & 0 \leq j \leq n \quad \text{and} \quad i = j, j+1 \\ \partial_n^{i-1} \varepsilon_{n-1}^j\,, & 0 \leq j \quad \text{and} \quad j+1 < i \leq n+1 \end{cases} \end{aligned} . \tag{3.3}$$

▽

Definition 3.1.4. Let C be a category. The category of *cosimplicial objects in* C is the functor category C^{Δ}, whilst the category of *simplicial objects in* C is the functor category $\mathsf{C}^{\Delta^{\mathrm{op}}}$, with Δ the simplex category. If $\mathsf{C} = \mathsf{Set}$ is the category of sets, then the category $\mathsf{Set}^{\Delta^{\mathrm{op}}}$ is denoted by sSet and we refer to it as the category of *simplicial sets.*

A simplicial set is a functor $S : \Delta^{\mathrm{op}} \to \mathsf{Set}$ and, thus, it associates each object $[n] \in \Delta$ with a set $S[n]$. We denote such a set by S_n and we refer to it as the set of n-simplices of S.

The functor S acts on face and degeneracy maps returning some distinguished maps between sets of simplices of S. Thus, we have face maps $\partial_i^n : S_n \to S_{n-1}$ and degeneracy maps $\varepsilon_i^n : S_n \to S_{n+1}$. Thanks to the functoriality of S, these maps fulfill the conditions dual to (3.3), which are called *simplicial identities*:

$$\begin{aligned}
\partial_i^n \partial_j^{n+1} &= \partial_{j-1}^n \partial_i^{n+1}\,, && 0 \le i < j \le n+1 \\
\varepsilon_i^{n+1} \varepsilon_j^n &= \varepsilon_j^{n+1} \varepsilon_{i-1}^n\,, && 0 \le j < i \le n+1 \\
\partial_i^{n+1} \varepsilon_j^n &= \begin{cases} \varepsilon_{j-1}^{n-1} \partial_i^n\,, & 0 \le i < j \le n \\ \mathrm{id}_n\,, & 0 \le j \le n \quad \text{and} \quad i = j, j+1 \\ \varepsilon_j^{n-1} \partial_{i-1}^n\,, & 0 \le j \quad \text{and} \quad j+1 < i \le n+1 \end{cases} .
\end{aligned} \tag{3.4}$$

Remark 3.1.5. A simplicial set S is equivalent to a collection of sets S_n together with maps ∂_i^n and ε_i^n as above satisfying the simplicial identities. ▽

Our aim is to associate a simplicial set to the field groupoid G^{LG}. To this end, we consider the nerve of the groupoid.

The nerve construction is rather technical and we are going to sketch it without the aim of exhausting the topic or of dwelling into the details. See [Mac98] for a broader presentation of this topic.

Let C be a category which comes endowed with a cosimplicial object $\Delta_{\mathsf{C}} : \Delta \to \mathsf{C}$. We use this functor to determine a realization of the standard n-simplex in C. As a matter of fact Δ_{C} induces a functor $|-| : \mathsf{sSet} \to \mathsf{C}$ which behaves like a geometric realization.

Explicitly, the realization functor $|-| : \mathsf{sSet} \to \mathsf{C}$ is the left Kan extension of Δ_{C} along the Yoneda embedding $Y : \Delta \hookrightarrow \mathsf{sSet}$.

$$\begin{array}{ccc} \Delta & \xrightarrow{\quad \Delta_{\mathsf{C}} \quad} & \mathsf{C} \\ & {\scriptstyle Y}\searrow \qquad \nearrow{\scriptstyle |-|} & \\ & \mathsf{sSet} & \end{array} \tag{3.5}$$

Moreover, the realization functor $|-|$ turns out to be the left adjoint in a pair of adjoint functors, $|-| \dashv N$, the right adjoint being the nerve functor $N : \mathsf{C} \to \mathsf{sSet}$.

The nerve functor N admits an explicit expression too. For each $c \in \mathsf{C}$, the contravariant functor $N(c)$ is given by the composition

$$N(c) : \Delta^{\mathrm{op}} \xrightarrow{\;\Delta_{\mathsf{C}}{}^{\mathrm{op}}\;} \mathsf{C}^{\mathrm{op}} \xrightarrow{\;\mathsf{C}(-,c)\;} \mathsf{Set}\,. \tag{3.6}$$

With this construction we manage to associate to each object $c \in \mathsf{C}$ a simplicial set $N(c)$. This simplicial set is referred to as the nerve of c with respect to the Δ_{C} functor.

We want to compute explicitly the nerve of G^{LG}. Here we follow closely [BSS15].

We take $\mathsf{C} = \mathsf{Cat}$, the category of small categories and functors between them. The role of the cosimplicial object Δ_{C} is played here by the inclusion

$$\Delta \xhookrightarrow{i} \mathsf{Cat}\,, \tag{3.7}$$

which embeds the simplex category as the full subcategory of Cat on non-empty finite totally ordered sets regarded as categories. This means that an object $[n] \in \Delta$ is identified with the category $\{0 \to 1 \to \cdots \to n\}$ and a morphism of Δ is identified with a functor between these totally ordered sets.

Let c be a small category, then its nerve is the simplicial set given by

$$N(c) : \Delta^{\mathrm{op}} \hookrightarrow \mathsf{Cat}^{\mathrm{op}} \xrightarrow{\mathsf{Cat}(-,c)} \mathsf{Set}\,. \tag{3.8}$$

Therefore, the set $N(c)_n$ of n-simplices of the nerve is the set of functors $\{0 \to 1 \to \cdots \to n\} \to c$. This is the same as the set of sequences of composable morphisms in c of length n. The face and degeneracy maps of the nerve are given by composing morphisms and inserting identities, respectively.

We can depict the nerve of the groupoid of linearized gravity, $N(\mathsf{G}^{LG})$, as

$$\Gamma(\otimes_S^2 T^*M) \Longleftarrow \Gamma(T^*M) \times \Gamma(\otimes_S^2 T^*M) \Lleftarrow \Gamma(T^*M)^{\times 2} \times \Gamma(\otimes_S^2 T^*M) \;\substack{\longleftarrow\\\longleftarrow\\\longleftarrow\\\longleftarrow}\; \cdots\,, \tag{3.9}$$

where the arrows are the face maps and the degeneracy maps are suppressed for notational clarity. We give an explicit expression for both face and degeneracy maps:

$$\begin{aligned} \partial_i^n : \Gamma(\otimes_S^2 T^*M) \times \Gamma(T^*M)^{\times n} &\longrightarrow \Gamma(\otimes_S^2 T^*M) \times \Gamma(T^*M)^{\times (n-1)}\,, \\ (h, \chi_1, \dots, \chi_n) &\longmapsto \begin{cases} (h + \nabla_S \chi_1, \chi_2, \dots, \chi_n)\,, & i = 0 \\ (h, \chi_1, \dots, \chi_i + \chi_{i+1}, \dots, \chi_n)\,, & 1 \leq i \leq n-1 \\ (h, \chi_1, \dots, \chi_{n-1})\,, & i = n \end{cases}\,, \end{aligned} \tag{3.10a}$$

and

$$\begin{aligned} \varepsilon_i^n : \Gamma(\otimes_S^2 T^*M) \times \Gamma(T^*M)^{\times n} &\longrightarrow \Gamma(\otimes_S^2 T^*M) \times \Gamma(T^*M)^{\times (n+1)}\,, \\ (h, \chi_1, \dots, \chi_n) &\longmapsto (h, \chi_1, \dots, \chi_i, 0, \chi_{i+1}, \dots \chi_n)\,. \end{aligned} \tag{3.10b}$$

Here we used the following notation to indicate the morphisms of G^{LG}: The morphism $h \xrightarrow{\chi} h + \nabla_S \chi$ is described by the pair $(h, \chi) \in \Gamma(\otimes_S^2 T^*M) \times \Gamma(T^*M)$. The notation is then extended to composable morphisms in the obvious way: $(h, \chi_1, \dots, \chi_n)$ denotes the chain of compositions $h \xrightarrow{\chi_1} h + \nabla_S \chi_1 \xrightarrow{\chi_2} \cdots \xrightarrow{\chi_n} h + \nabla_S \chi_1 + \cdots + \nabla_S \chi_n$.

Remark 3.1.6. This presentation of the content of the gauge field configurations is somehow more convenient than the groupoid perspective as in Equation (3.1). Indeed, it makes clear how to describe the observables for the gauge theory. If we interpret

Equation (3.9) as the simplicial set of gauge field configurations, then the observables, which have to test the field configurations, are naturally built as functions on it. This can be done by taking the degreewise algebra of functions on Equation (3.9). By dualizing the face and the degeneracy maps, *cf.* Equation (3.10), one finds a cosimplicial algebra of (classical) observables.

This perspective is related to the BV-BRST formalism, where the $\Gamma(T^*M)$ factors in the nerve (3.9) are interpreted as 'ghost fields'. For an introduction to BV-BRST formalism in the context of algebraic approach to field theory we refer the reader to [FR12a; FR12b]. ▽

This construction is rather general and the peculiarities of linearized gravity appear nowhere but in the very explicit expression of the nerve. We can now exploit some of the features of the theory of our interest to simplify the structures above.

Observe that all the sets in Equation (3.9) have a natural vector space structure inherited from the vector space structures on spaces of sections of vector bundles:

$$a(h,\chi_1,\dots,\chi_n)+b(h',\chi_1',\dots,\chi_n') \coloneqq (ah+bh', a\chi_1+b\chi_1',\dots,a\chi_n+b\chi_n')\,. \tag{3.11}$$

Moreover, the face and degeneracy maps of the nerve in Equation (3.10) are linear maps as one can see directly from their expressions. It follows that the simplicial set in Equation (3.9) is actually a simplicial vector space. This is very convenient because there exists an equivalence of categories between simplicial vector spaces and non-negatively graded chain complexes. This result goes under the name of Dold-Kan correspondence.

Theorem 3.1.7 (Dold-Kan correspondence)**.** *Let* $\mathsf{Vec}_{\mathbb{R}}$ *be the category of real vector spaces and let* $\mathsf{Ch}_{\mathbb{R}\geq 0}$ *be the one of non-negatively graded chain complexes of real vector spaces. Then, there is an adjoint equivalence of categories*

$$\mathcal{N} : \mathsf{Vec}_{\mathbb{R}}^{\Delta^{\mathrm{op}}} \rightleftarrows \mathsf{Ch}_{\mathbb{R}\geq 0} : \Gamma\ . \tag{3.12}$$

The functor $\mathcal{N}$ *is called the normalized Moore complex functor.*

We refer to [GJ12] for the proof of this theorem.

At the moment, we are only interested in the existence of such an equivalence and in particular we need an explicit description of the functor $\mathcal{N}$. From the proof one finds the following: Let $V=(V_n)_{n\in\mathbb{N}}$ be any simplicial vector space with face and degeneracy maps $\partial_i^n : V_n \to V_{n-1}$ and $\varepsilon_i^n : V_n \to V_{n+1}$. Then the functor $\mathcal{N}$ yields the chain complex $\mathcal{N}(V)$ which in degree zero is simply V_0 while in degree $n\geq 1$ is the quotient between V_n and the image of the degeneracy maps: $\mathcal{N}(V)_n \coloneqq V_n/\varepsilon_0^{n-1}(V_{n-1})+\dots+\varepsilon_{n-1}^{n-1}(V_{n-1})$. The differential d of the complex is the alternating sum of the face maps. Explicitly, we set

$$\mathrm{d} \coloneqq \sum_{i=0}^{n} (-1)^i \partial_i^n \tag{3.13}$$

on $\mathcal{N}(V)_n$.

Corollary 3.1.8. *Let G^{LG} be the groupoid of linearized gravity as per Equation (3.1). The normalized Moore complex associated with the nerve $N(\mathsf{G}^{LG})$ is*

$$\mathfrak{C}(M) = \Big(\overset{(0)}{\Gamma(\otimes_S^2 T^*M)} \xleftarrow{\nabla_S} \overset{(1)}{\Gamma(T^*M)} \Big) , \tag{3.14}$$

where the round brackets indicate the homological degrees.

Proof. We have to follow the procedure described above and we refer to Equations (3.9) and (3.10) for the form of the simplicial set $N(\mathsf{G}^{LG})$.
The zeroth homological degree of $\mathfrak{C}(M)$ is by definition $N(\mathsf{G}^{LG})_0 = \Gamma(\otimes_S^2 T^*M)$.
The first homology degree is

$$\mathfrak{C}_1(M) = \Gamma(\otimes_S^2 T^*M) \times \Gamma(T^*M)/\varepsilon_0^0(\Gamma(\otimes_S^2 T^*M)) ; \tag{3.15}$$

The image $\varepsilon_0^0(\Gamma(\otimes_S^2 T^*M))$ is $\Gamma(\otimes_S^2 T^*M) \times 0$; Therefore $\mathfrak{C}_1(M) \cong \Gamma(T^*M)$.

All the remaining degrees of the chain complex are trivial. Indeed,

$$\varepsilon_i^{n-1}(N(\mathsf{G}^{LG})_{n-1}) = \Gamma(\otimes_S^2 T^*M) \times \underbrace{\Gamma(T^*M) \times \cdots \times \overbrace{0}^{(i+1)\text{-th}} \times \cdots \times \Gamma(T^*M)}_{n} , \tag{3.16}$$

hence, $\Gamma(\otimes_S^2 T^*M) \times \Gamma(T^*M)^{\times n} = \varepsilon_0^{n-1}(N(\mathsf{G}^{LG})_{n-1}) + \cdots + \varepsilon_{n-1}^{n-1}(N(\mathsf{G}^{LG})_{n-1})$.

Finally, we need to compute the differential $\mathrm{d} : \mathfrak{C}_1(M) \to \mathfrak{C}_0(M)$. Let $\chi \in \Gamma(T^*M)$, then the differential as per Equation (3.13) reads explicitly

$$\mathrm{d}\,\chi = \partial_0^1(\chi, 0) - \partial_1^1(\chi, 0) = \nabla_S \chi . \tag{3.17}$$

This concludes the proof. □

We refer to the complex in Equation (3.14) as the *linearized gravity field complex* on M. Let us comment on the physical content of such complex. The elements in degree 0 are the gauge fields $h \in \Gamma(\otimes_S^2 T^*M)$ and those in degree 1 are interpreted as gauge transformations $\chi \in \Gamma(T^*M)$. The differential encodes the action of gauge transformations on gauge fields, *i.e.* $\chi : h \to h + \nabla_S \chi$.

3.2 The dynamics

The linearized gravity field complex $\mathfrak{C}(M)$ from the previous section carries all the information regarding the structure of off-shell configurations of linearized gravity. Up to this point, no information about the dynamics has been taken into account. Indeed, the linearized Einstein's equations (2.9) did not come into play at any level of our construction. In other words, the complex in Equation (3.14) contains only the kinematical and gauge structures of the fields.

Therefore, in this section we want to implement the dynamics in a suitable way. We recall here that the category $\mathsf{Ch}_{\mathbb{R}}$ of chain complexes and chain maps can be endowed with a model structure whose weak equivalences are quasi-isomorphisms. Complying with the model category philosophy, weakly equivalent complexes are regarded as being the same. This applies also to the way the dynamics is encoded

We are going to establish an analogue of the principle of least action that works properly in this framework.

Let us start by defining a quadratic action functional associated with the linearized Einstein's equations. We recall that they are written in terms of a formally self-adjoint operator

$$P : \mathfrak{C}_0(M) \longrightarrow \mathfrak{C}_0(M)\,, \quad h \longmapsto Ph = (-\Box + 2\,\mathrm{Riem} + 2I\nabla_S\,\mathrm{div})\,Ih\,. \tag{3.18}$$

Moreover, both the degrees 0 and 1 of the field complex in Equation (3.14) are endowed with pairings which are given by the integral pairing as per Equation (2.28). For convenience, we write out explicitly these pairings:

$$(-,-) : \mathfrak{C}_0(M)\check{\times}\mathfrak{C}_0(M) \longrightarrow \mathbb{R}\,, \qquad (h,u) \coloneqq \int_M h_{ab}u_{cd}g^{ac}g^{bd}\mu_g\,, \tag{3.19a}$$

$$(-,-) : \mathfrak{C}_1(M)\check{\times}\mathfrak{C}_1(M) \longrightarrow \mathbb{R}\,, \qquad (\chi,\eta) \coloneqq \int_M \chi_a\eta_b g^{ab}\mu_g\,. \tag{3.19b}$$

A quadratic action functional is constructed via both the dynamical operator P and the integral pairing in Equation (3.19a).

Definition 3.2.1. The quadratic *action functional* for linearized gravity is formally defined as

$$\begin{aligned} S : \mathfrak{C}_0(M) &\longrightarrow \mathbb{R} \\ h &\longmapsto S(h) \coloneqq \frac{1}{2}(h,Ph) = \frac{1}{2}\int_M h_{ab}(Ph)_{cd}g^{ac}g^{bd}\mu_g\,. \end{aligned} \tag{3.20}$$

Remark 3.2.2. Observe that the action functional S extends naturally to a chain map from the field complex to $\mathbb{R}$, seen has a chain complex concentrated in degree 0. This is a direct consequence of the gauge invariance of the dynamical operator P. Indeed, we need to check the commutativity of the following diagram

$$\begin{array}{ccccc} 0 & \longleftarrow & \Gamma(\otimes_S^2 T^*M) & \xleftarrow{\nabla_S} & \Gamma(T^*M) \\ \downarrow & & \downarrow S & & \downarrow \\ 0 & \longleftarrow & \mathbb{R} & \longleftarrow & 0 \end{array} \tag{3.21}$$

The only non straightforward check concerns the right square. Yet

$$S(\nabla_S\chi) = \frac{1}{2}(\nabla_S\chi, P\nabla_S\chi) = 0\,, \quad \forall\chi \in \Gamma(T^*M)\,, \tag{3.22}$$

since $\nabla_S\Gamma(T^*M) \subseteq \mathrm{Ker}\,P$, see Theorem 2.1.14. $\triangledown$

To formulate a principle of least action we need to write a variation δS of the action functional. Therefore, let $h \in \Gamma(\otimes^2_S T^*M)$, $u \in \Gamma_c(\otimes^2_S T^*M)$ and $\lambda \in \mathbb{R}$. We are working formally here by choosing a compactly supported section u: This ensures the finiteness of all quantities in hand. We compute

$$\begin{aligned} S(h+\lambda u) - S(h) &= \frac{1}{2}(h+\lambda u, P(h+\lambda u)) - S(h) \\ &= \frac{1}{2}\left\{(h,Ph) + \lambda\left[(h,Pu)+(u,Ph)\right] + \lambda^2(u,Pu)\right\} - S(h) \\ &= \lambda(Ph,u) + \frac{\lambda^2}{2}(u,Pu)\,, \end{aligned} \tag{3.23}$$

where the last step follows from the formal self-adjointness of P and the fact that the pairing is symmetrical. Therefore, by taking the limit for $\lambda \to 0$ of the difference quotient, we get

$$\frac{S(h+\lambda u) - S(h)}{\lambda} \xrightarrow{\lambda \to 0} (Ph,u) =: \delta S_h(u)\,. \tag{3.24}$$

The functional $\delta S_h : \Gamma_c(\otimes^2_S T^*M) \to \mathbb{R}$, $\delta S_h = (Ph, -)$ is analogous to the Gâteaux derivative of the action S along the direction of the field h. Since the pairing $(-,-)$ is non-degenerate, it is possible to identify the functional δS_h with the smooth section $Ph \in \Gamma(\otimes^2_S T^*M)$.

Therefore, we can interpret δS as a section of the trivial vector bundle $\Gamma(\otimes^2_S T^*M) \times \Gamma(\otimes^2_S T^*M) \xrightarrow{\pi_1} \Gamma(\otimes^2_S T^*M)$:

$$\delta S : \Gamma(\otimes^2_S T^*M) \longrightarrow \Gamma(\otimes^2_S T^*M) \times \Gamma(\otimes^2_S T^*M), \quad \delta S(h) \coloneqq (h, Ph)\,. \tag{3.25}$$

So far we have considered the action functional simply as a map from $\Gamma(\otimes^2_S T^*M)$ to $\mathbb{R}$. Now we need to extend our construction taking into account the chain complex structure of the field configurations and the fact that S is naturally a chain map, as stated in Remark 3.2.2.

The first step consists of giving a notion of cotangent bundle for the field complex $\mathfrak{C}(M)$. A reasonable choice is to mimic the structure of a trivial cotangent bundle. Thence we define the following *cotangent bundle complex*

$$T^*\mathfrak{C}(M) \coloneqq \mathfrak{C}(M) \times \mathfrak{C}_c(M)^*\,, \tag{3.26}$$

where $\mathfrak{C}_c(M)^*$ is a smooth dual of the compactly supported version of the linearized gravity field complex. This smooth dual is built with the idea of constructing a chain complex which is dually paired with $\mathfrak{C}_c(M)$ with respect to a pairing which is given by the integrals in Equations (3.19). To be explicit, this pairing is defined as the chain map

$$(-,-) : \mathfrak{C}_c(M)^* \otimes \mathfrak{C}_c(M) \longrightarrow \overset{(0)}{\mathbb{R}}\,, \tag{3.27}$$

which pairs the proper section spaces coherently with the already given integral pairings.

We set

$$\mathfrak{C}_c(M)^* := \Big(\overset{(-1)}{\Gamma(T^*M)} \xleftarrow{\operatorname{div}} \overset{(0)}{\Gamma(\otimes_S^2 T^*M)} \Big). \tag{3.28}$$

Observe that the homological degrees are flipped with respect to those of the original complex $\mathfrak{C}_c(M)$. The tensor product $\mathfrak{C}_c(M)^* \otimes \mathfrak{C}_c(M)$ has the following non-vanishing degrees

$$\begin{aligned} (\mathfrak{C}_c(M)^* \otimes \mathfrak{C}_c(M))_{-1} &= \Gamma(T^*M) \otimes \Gamma_c(\otimes_S^2 T^*M) \\ (\mathfrak{C}_c(M)^* \otimes \mathfrak{C}_c(M))_0 &= (\Gamma(T^*M) \otimes \Gamma_c(T^*M)) \oplus \big(\Gamma(\otimes_S^2 T^*M) \otimes \Gamma_c(\otimes_S^2 T^*M)\big) \\ (\mathfrak{C}_c(M)^* \otimes \mathfrak{C}_c(M))_1 &= \Gamma(\otimes_S^2 T^*M) \otimes \Gamma_c(T^*M)\,. \end{aligned} \tag{3.29}$$

Moreover, the differentials are given by the graded Leibniz rule, as per Equation (1.13), and they read explicitly:

$$\begin{aligned} \mathrm{d} : (\mathfrak{C}_c(M)^* \otimes \mathfrak{C}_c(M))_1 &\longrightarrow (\mathfrak{C}_c(M)^* \otimes \mathfrak{C}_c(M))_0\,, \\ \mathrm{d}\,(h \otimes \chi) &:= \operatorname{div} h \otimes \chi + h \otimes \nabla_S \chi\,, \end{aligned} \tag{3.30}$$

and

$$\begin{aligned} \mathrm{d} : (\mathfrak{C}_c(M)^* \otimes \mathfrak{C}_c(M))_0 &\longrightarrow (\mathfrak{C}_c(M)^* \otimes \mathfrak{C}_c(M))_{-1}\,, \\ \mathrm{d}\,(\chi \otimes \eta + h \otimes u) &:= -\chi \otimes \nabla_S \eta + \operatorname{div} h \otimes u\,. \end{aligned} \tag{3.31}$$

A direct calculation shows that $(-,-)$ is a chain map. This amounts to checking commutativity of the diagram below

$$\begin{array}{ccc} (\Gamma(T^*M) \otimes \Gamma_c(T^*M)) \oplus \big(\Gamma(\otimes_S^2 T^*M) \otimes \Gamma_c(\otimes_S^2 T^*M)\big) & \xleftarrow{\operatorname{div} \otimes \mathrm{id} \oplus \mathrm{id} \otimes \nabla_S} & \Gamma(\otimes_S^2 T^*M) \otimes \Gamma_c(T^*M) \\ \Big\downarrow{\scriptstyle (-,-)} & & \Big\downarrow \\ \mathbb{R} & \longleftarrow & 0 \end{array}$$

Let us check the commutativity condition: For each homogeneous element $h \otimes \chi \in \Gamma(\otimes_S^2 T^*M) \otimes \Gamma_c(T^*M)$, we have

$$(\operatorname{div} h, \chi) + (h, \nabla_S \chi) = (\operatorname{div} h, \chi) + (-\operatorname{div} h, \chi) = 0\,, \tag{3.32}$$

where in the last step we used Lemma 2.2.4.

With this notion of smooth dual complex we are finally able to compute the cotangent bundle complex (3.26).

Lemma 3.2.3. *Let* $\mathfrak{C}(M) = \Big(\Gamma(\otimes_S^2 T^*M) \xleftarrow{\nabla_S} \Gamma(T^*M) \Big)$ *be the linearized gravity field complex. Then, its cotangent bundle complex reads explicitly as*

$$T^*\mathfrak{C}(M) = \Big(\overset{(-1)}{\Gamma(T^*M)} \xleftarrow{\operatorname{div} \pi_2} \overset{(0)}{\Gamma(\otimes_S^2 T^*M) \times \Gamma(\otimes_S^2 T^*M)} \xleftarrow{\iota_1 \nabla_S} \overset{(1)}{\Gamma(T^*M)} \Big), \tag{3.33}$$

where $\iota_1 : \Gamma(\otimes_S^2 T^*M) \to \Gamma(\otimes_S^2 T^*M) \times \Gamma(\otimes_S^2 T^*M)$ *is the inclusion in the first factor, while* $\pi_2 : \Gamma(\otimes_S^2 T^*M) \times \Gamma(\otimes_S^2 T^*M) \to \Gamma(\otimes_S^2 T^*M)$ *is the projection onto the second one. As usual, the round brackets indicate the homological degrees.*

Proof. We need to compute the product of the *base* $\mathfrak{C}(M)$ and the *fibre* $\mathfrak{C}_c(M)^*$:

$$T^*\mathfrak{C}(M) = \left(\overset{(0)}{\Gamma(\otimes_S^2 T^*M)} \xleftarrow{\nabla_S} \overset{(1)}{\Gamma(T^*M)} \right) \times \left(\overset{(-1)}{\Gamma(T^*M)} \xleftarrow{\mathrm{div}} \overset{(0)}{\Gamma(\otimes_S^2 T^*M)} \right) . \tag{3.34}$$

The product is computed degreewise:

$$\begin{gathered} (T^*\mathfrak{C}(M))_n = \mathfrak{C}_n(M) \times \mathfrak{C}_{c,n}(M)^* , \\ \mathrm{d}_n : (T^*\mathfrak{C}(M))_n \longrightarrow (T^*\mathfrak{C}(M))_{n-1} , \quad \mathrm{d}_n(u,v) = (\mathrm{d}\, u, \mathrm{d}\, v) . \end{gathered} \tag{3.35}$$

To be explicit, we find the following non trivial degrees:

- degree 1: $\Gamma(T^*M)$;
- degree 0: $\Gamma(\otimes_S^2 T^*M) \times \Gamma(\otimes_S^2 T^*M)$;
- degree -1: $\Gamma(T^*M)$.

With reference to the differentials, we get $\mathrm{d}_1\, \chi = (\nabla_S\chi, 0) = \iota_1\nabla_S\chi$ for the one from degree 1 to degree 0, and $\mathrm{d}_0\,(h,w) = (0, \mathrm{div}\, w) = \mathrm{div}\, \pi_2(h,w)$ for the one from degree 0 to degree -1. □

Since now we have a model for the cotangent bundle, a suitable variation of the action chain map can be presented. This will be a section $\delta S : \mathfrak{C}(M) \to T^*\mathfrak{C}(M)$. With *section* here we mean that the composition $\pi_1 \circ \delta S$ coincides with the identity chain map id, where π_1 is the projection onto the first factor. The actual definition of this section is

$$\begin{array}{c} \mathfrak{C}(M) \\ \delta S \big\downarrow \\ T^*\mathfrak{C}(M) \end{array} := \left(\begin{array}{ccccc} 0 & \longleftarrow & \Gamma(\otimes_S^2 T^*M) & \xleftarrow{\nabla_S} & \Gamma(T^*M) \\ \big\downarrow & & \big\downarrow{\scriptstyle (\mathrm{id},P)} & & \big\downarrow{\scriptstyle \mathrm{id}} \\ \Gamma(T^*M) & \xleftarrow[\mathrm{div}\,\pi_2]{} & \Gamma(\otimes_S^2 T^*M) \times \Gamma(\otimes_S^2 T^*M) & \xleftarrow[\iota_1\nabla_S]{} & \Gamma(T^*M) \end{array} \right) . \tag{3.36}$$

Observe that this map is obtained by extending the variation of the action S, as per Equation (3.25), consistently with the section condition.

Remark 3.2.4. It is easy to check that δS is a chain map. Indeed, the commutativity of the previous diagram is a consequence of the gauge invariance of the dynamical operator P.

The left square commutes because $\mathrm{div}\, \pi_2(\mathrm{id}, P)h = \mathrm{div}\,(Ph) = 0$, where the last identity is the dual of $P\nabla_S h = 0$, which is stated in Proposition 2.1.14. The right square commutes because $(\mathrm{id}, P)\nabla_S\chi = (\nabla_S\chi, P(\nabla_S\chi)) = (\nabla_S\chi, 0) = \iota_1\nabla_S\chi$. ▽

To find the critical points of the action S, we need to intersect the variation of the action δS with the zero-section $0 : \mathfrak{C}(M) \to T^*\mathfrak{C}(M)$ of the cotangent bundle.

The zero-section is defined as the chain map

$$\begin{array}{c}\mathfrak{C}(M)\\ {\scriptstyle 0}\big\downarrow\\ T^*\mathfrak{C}(M)\end{array} := \left(\begin{array}{ccccc} 0 & \longleftarrow & \Gamma(\otimes^2_S T^*M) & \xleftarrow{\nabla_S} & \Gamma(T^*M) \\ \big\downarrow & & {\scriptstyle (\mathrm{id},0)}\big\downarrow & & {\scriptstyle \mathrm{id}}\big\downarrow \\ \Gamma(T^*M) & \xleftarrow[\mathrm{div}\,\pi_2]{} & \Gamma(\otimes^2_S T^*M)\times\Gamma(\otimes^2_S T^*M) & \xleftarrow[\iota_1\nabla_S]{} & \Gamma(T^*M)\end{array}\right). \tag{3.37}$$

Finding the critical points amount to taking a pullback of our two chain maps. This is formalized in the following definition.

Definition 3.2.5 (Linearized gravity solution complex)**.** Let $\mathfrak{C}(M)$ be the linearized gravity field complex on a physical spacetime M and let $T^*\mathfrak{C}(M)$ be its cotangent bundle. Then let $\delta S : \mathfrak{C}(M) \to T^*\mathfrak{C}(M)$ be the variation of the action chain map, built with the dynamical operator in Equation (2.10). The corresponding *solution complex* is defined as the homotopy pullback

$$\begin{array}{ccc} \mathfrak{Sol}(M) & \dashrightarrow & \mathfrak{C}(M) \\ \big\downarrow & h & \big\downarrow{\scriptstyle \delta S} \\ \mathfrak{C}(M) & \xrightarrow[0]{} & T^*\mathfrak{C}(M) \end{array} \tag{3.38}$$

in the model category $\mathsf{Ch}_{\mathbb{R}}$.

Remark 3.2.6. A pullback is the categorical semantics of an equation, hence it is natural to come up with this notion if we want to find critical points of the action. However, the ordinary pullback is not the appropriate choice. Indeed, the category of chain complexes is endowed with the structure of a model category and then we need to consider quasi-isomorphic chain complexes as the same. Unfortunately, the ordinary pullback does not preserve weak-equivalences, *cf.* [Hov07]. This means that replacing $\mathfrak{C}(M)$ by a quasi-isomorphic chain complex may not yield a quasi-isomorphic solution complex via an ordinary categorical pullback.

This is the reason why Definition 3.2.5 is given in terms of an homotopy pullback in the model category $\mathsf{Ch}_{\mathbb{R}}$. This is the best approximation, technically a derived functor, of the ordinary pullback which preserves weak-equivalences. As a consequence, the solution complex $\mathfrak{Sol}(M)$ that we have defined encodes the equation of motion only in a weak sense, that is "up to homotopy". ▽

We need now to compute the homotopy pullback in order to get the explicit form of the linearized gravity solution complex. This is exactly the content of the following proposition.

Proposition 3.2.7. *The linearized gravity solution complex of Definition 3.2.5 is given explicitly by*

$$\mathfrak{Sol}(M) = \Big(\overset{(-2)}{\Gamma(T^*M)} \xleftarrow{-\mathrm{div}} \overset{(-1)}{\Gamma(\otimes^2_S T^*M)} \xleftarrow{P} \overset{(0)}{\Gamma(\otimes^2_S T^*M)} \xleftarrow{\nabla_S} \overset{(1)}{\Gamma(T^*M)} \Big). \tag{3.39}$$

Proof. We start by recalling that $\mathsf{Ch}_{\mathbb{R}}$ is a (right) proper category. This implies that any homotopy pullback can be calculated by taking the ordinary pullback of a diagram obtained by replacing one of the morphisms by a weakly equivalent fibration. We refer the reader to [Hir09] for all the details.

Therefore, we look for a complex Z, a quasi-isomorphism $\sim$ and a fibration $\widetilde{0}$ such that

$$\begin{array}{ccc} \mathfrak{C}(M) & \xrightarrow{\quad 0 \quad} & T^*\mathfrak{C}(M) \\ & {\scriptstyle\sim}\searrow \quad Z \quad \nearrow{\scriptstyle\widetilde{0}} & \end{array} . \tag{3.40}$$

Observe that the zero-section in Equation (3.37) is defined as the product of the identity map $\mathrm{id} : \mathfrak{C}(M) \to \mathfrak{C}(M)$ and the zero map $0 : 0 \to \mathfrak{C}_c(M)^*$. Our problem reduces thus to finding a fibration that replaces $0 : 0 \to \mathfrak{C}_c(M)^*$.

Let us introduce the chain complex $D := \left(\overset{(-1)}{\mathbb{R}} \xleftarrow{\ \mathrm{id}\ } \overset{(0)}{\mathbb{R}} \right)$. This is weakly equivalent to the 0 complex since all its homologies are trivial. See Lemma 1.1.12.

We can use this to factor the map $0 \to \mathbb{R}$, where $\mathbb{R}$ is concentrated in degree 0. To be very explicit, we draw the following triangle

$$\begin{array}{ccc} 0 & \xrightarrow{\qquad} & \mathbb{R} \\ & {\scriptstyle\sim}\searrow \quad D \quad \nearrow{\scriptstyle\iota} & \end{array} , \tag{3.41}$$

where the map $\iota : D \to \mathbb{R}$ is given by

$$\begin{array}{c} D \\ {\scriptstyle\iota}\downarrow \\ \mathbb{R} \end{array} = \left(\begin{array}{ccc} \mathbb{R} & \xleftarrow{\ \mathrm{id}\ } & \mathbb{R} \\ \downarrow & & \downarrow{\scriptstyle\mathrm{id}} \\ 0 & \longleftarrow & \mathbb{R} \end{array} \right) \tag{3.42}$$

and it is manifestly a fibration.

Let us take the tensor product of the factorization in Equation (3.41) with $\mathfrak{C}_c(M)^*$. We get the following factorization of the map $0 \to \mathfrak{C}_c(M)^*$ into a weak equivalence followed by a fibration:

$$0 \xrightarrow{\ \sim\ } D \otimes \mathfrak{C}_c(M)^* \xrightarrow{\ \iota\otimes\mathrm{id}\ } \mathbb{R} \otimes \mathfrak{C}_c(M)^* \cong \mathfrak{C}_c(M)^* \,. \tag{3.43}$$

Finally, we need to take the product with $\mathfrak{C}(M)$ of the factorization above in order to get the triangle in Equation (3.40). Referring to this latter diagram, we have $Z = \mathfrak{C}(M) \times (D \otimes \mathfrak{C}_c(M)^*)$ and $\widetilde{0} = \mathrm{id} \times (\iota \otimes \mathrm{id})$. Chain map $\widetilde{0}$ is a fibration since it is surjective in each degree and $\mathfrak{C}(M) \xrightarrow{\sim} Z$ is a weak equivalence since it is the identity at the level of homology. With some straightforward calculations, we can explicitly write them:

$$Z = \Big(\overset{(-2)}{\Gamma(T^*M)} \xleftarrow{-\mathrm{div}\,\pi_1+\pi_2} \overset{(-1)}{\Gamma(\otimes^2_S T^*M) \times \Gamma(T^*M)} \xleftarrow{(\mathrm{id},\mathrm{div})\pi_2} \overset{(0)}{\Gamma(\otimes^2_S T^*M) \times \Gamma(\otimes^2_S T^*M)} \xleftarrow{\iota_1\nabla_S} \overset{(1)}{\Gamma(T^*M)} \Big), \tag{3.44}$$

where the fibration $\widetilde{0}$ is the identity in degrees 0 and 1 and it is the projection π_2 onto the second factor in the degree -1.

Therefore, the homotopy pullback of the diagram $\mathfrak{C}(M) \xrightarrow{0} T^*\mathfrak{C}(M) \xleftarrow{\delta S} \mathfrak{C}(M)$ is (weakly equivalent to) the ordinary pullback of $Z \xrightarrow{\widetilde{0}} T^*\mathfrak{C}(M) \xleftarrow{\delta S} \mathfrak{C}(M)$.

The pullback can be computed degreewise by taking the intersection between the two morphisms $\widetilde{0}$ and δS. More concretely, we have to take the chain subcomplex of $Z \times \mathfrak{C}(M)$ such that (the extensions of) the two morphisms coincide. The following diagram illustrates the situation:

$$\begin{array}{ccccccccc}
Z \times \mathfrak{C}(M) & & \Gamma(T^*M) & \xleftarrow{-\operatorname{div}\pi_1+\pi_2} & \Gamma(\otimes^2_S T^*M) \times \Gamma(T^*M) & \xleftarrow{(\mathrm{id},\operatorname{div})\pi_2} & \Gamma(\otimes^2_S T^*M)^{\times 3} & \xleftarrow{(\nabla_S\pi_1,0,\nabla_S\pi_2)} & \Gamma(T^*M)^{\times 2} \\
\widetilde{0}\Big\downarrow = \Big\downarrow\delta S & = & \Big\downarrow = \Big\downarrow & & \pi_2\Big\downarrow = \Big\downarrow 0 & & (\pi_1,\pi_2)\Big\downarrow = \Big\downarrow(\mathrm{id},P)\pi_3 & & \pi_1\Big\downarrow = \Big\downarrow\pi_2 \\
T^*\mathfrak{C}(M) & & 0 & \longleftarrow & \Gamma(T^*M) & \xleftarrow[\operatorname{div}\pi_2]{} & \Gamma(\otimes^2_S T^*M)^{\times 2} & \xleftarrow[\iota_1\nabla_S]{} & \Gamma(T^*M)
\end{array}$$

By calculating, degree by degree, the subspaces which fulfill the required equalities and the induced differentials, we find

$$Z \times_{T^*\mathfrak{C}(M)} \mathfrak{C}(M) = \left(\overset{(-2)}{\Gamma(T^*M)} \xleftarrow{-\operatorname{div}} \overset{(-1)}{\Gamma(\otimes^2_S T^*M)} \xleftarrow{P} \overset{(0)}{\Gamma(\otimes^2_S T^*M)} \xleftarrow{\nabla_S} \overset{(1)}{\Gamma(T^*M)} \right), \tag{3.45}$$

namely the stated solution complex. □

Remark 3.2.8. Let us go back, for a moment, to the necessity of imposing the equation of motion only in a weak sense. If we had enforced it in a strong sense, *i.e.* with the ordinary pullback of $\mathfrak{C}(M) \xrightarrow{0} T^*\mathfrak{C}(M) \xleftarrow{\delta S} \mathfrak{C}(M)$, we would have found the chain complex

$$\mathfrak{Sol}_{\mathrm{str}}(M) = \left(\overset{(0)}{\operatorname{Ker} P} \xleftarrow{\nabla_S} \overset{(1)}{\Gamma(T^*M)} \right). \tag{3.46}$$

This is a truncation of our "weak" solution complex as per Equation (3.39). Observe that in this case the vector space in degree 0 contains those sections h that solve the linearized Einstein's equations (2.8), while the vector space in degree 1 contains the gauge transformations.

It is also interesting to consider its homologies:

- $H_1(\mathfrak{Sol}_{\mathrm{str}}(M)) = \operatorname{Ker} \nabla_S$ describes those gauge transformations that act trivially on gauge fields;
- $H_0(\mathfrak{Sol}_{\mathrm{str}}(M)) = \operatorname{Ker} P / \operatorname{Im} \nabla_S$ is the usual space of on-shell gauge equivalence classes that we denoted previously as $\mathcal{C}_{\mathrm{on}}(M)$.

▽

We can return now to consider the linearized gravity solution complex $\mathfrak{Sol}(M)$. We can give a physical interpretation of the components of $\mathfrak{Sol}(M)$ in terms of the BV-BRST formalism:

- the fields in degree 0 are the gauge fields $h \in \Gamma(\otimes_S^2 T^*M)$;
- the fields in degree 1 are the ghost fields $\chi \in \Gamma(T^*M)$;
- the fields in degrees -1 and -2 are interpreted as the antifields $h^\ddagger \in \Gamma(\otimes_S^2 T^*M)$ and $\chi^\ddagger \in \Gamma(T^*M)$ of the gauge fields h and the ghost fields χ respectively.

Finally, it is of interest to consider the homologies of $\mathfrak{Sol}(M)$ and to try to compute them explicitly. We find the following:

- $H_1(\mathfrak{Sol}(M)) = \mathrm{Ker}\,\nabla_S$ describes again those gauge transformations that act trivially on gauge fields. Observe that $\mathrm{Ker}\,\nabla_S = \{\chi \in \Gamma(T^*M) \mid \nabla_a\chi_b + \nabla_b\chi_a = 0\}$ is the vector space of Killing vector fields (up to a musical isomorphism) on the Lorentzian manifold (M, g). Therefore, this homology conveys geometrical information about the isometries of the background spacetime. It is known from general results that this is a finite dimensional vector space. In particular, for a connected manifold with dimension equals to n the Killing vector fields space has dimension at most $n(n+1)/2$. See for example [ONe83, Lemma 28];
- $H_0(\mathfrak{Sol}(M)) = \mathrm{Ker}\,P/\,\mathrm{Im}\,\nabla_S = \mathcal{C}_{\mathrm{on}}(M)$ is the usual vector space of gauge equivalence classes of linearized gravity solutions;
- $H_{-1}(\mathfrak{Sol}(M)) = \mathrm{Ker}\,(\mathrm{div})/\,\mathrm{Im}\,P$ captures obstructions to solving the inhomogeneous linearized gravity equation $Ph = t$ with $t \in \Gamma(\otimes_S^2 T^*M)$ such that $\mathrm{div}\,t = 0$;
- $H_{-2}(\mathfrak{Sol}(M)) = \Gamma(T^*M)/\,\mathrm{Im}\,(\mathrm{div}) \cong 0$ is always trivial. This is proved in the following lemma.

Lemma 3.2.9. *Let M be a physical spacetime. Then, the divergence operator* $\mathrm{div} : \Gamma(\otimes_S^2 T^*M) \to \Gamma(T^*M)$ *is onto.*

Proof. We recall that $\Gamma_{pc/fc}(F)$ denotes the space of sections of a finite-rank vector bundle F with past compact and future compact support, respectively.

We will show that there exist maps $\lambda^\pm : \Gamma_{pc/fc}(T^*M) \to \Gamma_{pc/fc}(\otimes_S^2 T^*M)$ such that it holds $\mathrm{id}_{\Gamma_{pc/fc}(T^*M)} = \mathrm{div}\,\lambda^\pm$. This will be sufficient to prove the claim. Indeed, let $\chi \in \Gamma(T^*M)$ be any section. Consider a partition of unity $\{f_+, f_-\}$ on M, such that $f_\pm = 1$ on a past/future compact domain. Using the property $f_+ + f_- = 1$, one gets

$$\chi = f_+\chi + f_-\chi = \mathrm{div}\,(\lambda^+(f_+\chi)) + \mathrm{div}\,(\lambda^-(f_-\chi)) = \mathrm{div}\,(\lambda^+(f_+\chi) + \lambda^-(f_-\chi))\,, \quad (3.47)$$

thus proving that χ is in the image of the divergence operator.

Let us construct these maps. Let $G_\pm$ be the retarded/advanced Green operators for the differential operator P', we introduced in Section 2.2. They admit an extension to past/future compact sections. We denote these extensions with the same symbol,

$$G_\pm : \Gamma_{pc/fc}(\otimes_S^2 T^*M) \longrightarrow \Gamma_{pc/fc}(\otimes_S^2 T^*M)\,. \quad (3.48)$$

These operators have the usual support properties and it holds $P' \circ G_\pm = G_\pm \circ P' = \mathrm{id}_{\Gamma_{pc/fc}(\otimes^2_S T^*M)}$. A similar extension $G^\Box_\pm : \Gamma_{pc/fc}(T^*M) \to \Gamma_{pc/fc}(T^*M)$ holds true for the Green operators of $\Box$.

We define

$$\lambda^\pm := -2G_\pm \circ \nabla_S\,. \tag{3.49}$$

We need to check only if the sought identity holds:

$$\operatorname{div}\lambda^\pm = -2\operatorname{div}G_\pm\nabla_S = -2\operatorname{div}IG_\pm I\nabla_S = 2G^\Box_\pm\operatorname{div}I\nabla_S = G^\Box_\pm\Box = \mathrm{id}_{\Gamma_{pc/fc}(T^*M)}\,, \tag{3.50}$$

where Proposition 2.1.11 and definition of $G_\pm$ are used in the second step, Lemma 2.2.5 in the third step and the last identity in Lemma 2.1.13 in the fourth one. Since we exhibited explicitly the maps $\lambda^\pm$, the proof is complete. □

Remark 3.2.10. Observe that the complex $\mathfrak{Sol}(M)$ is not always weakly equivalent to the chain complex $\mathcal{C}_{\mathrm{on}}(M)$ concentrated in degree 0. This means that it contains more refined information than the on-shell gauge orbit vector space considered in Chapter 2.

Let us take, for simplicity, a background manifold M connected and of constant (sectional) curvature. In this case we can say something more about the homologies. A constant curvature manifold is also maximally symmetric, as it is shown in [Eis97]. Hence $H_1(\mathfrak{Sol}(M)) = \operatorname{Ker}\nabla_S \cong \mathbb{R}^{10}$, if M is simply connected and $\dim M = 4$. Moreover, the computation of $H_{-1}(\mathfrak{Sol}(M))$ can be reduced to that of the cohomology of the Calabi complex, or rather of its formal adjoint, which have been studied in [Kha17]. The part of the Calabi complex we are interested in reads explicitly:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(T^*M) & \xrightarrow{2\nabla_S} & \Gamma(\otimes^2_S T^*M) & \xrightarrow{B} & \Gamma(\otimes^4_{\mathcal{R}} T^*M) & \longrightarrow & \cdots \\
 & & \Box\downarrow & \overset{\operatorname{div} I}{\swarrow} & \Box\downarrow & \overset{\mathrm{tr}'}{\swarrow} & \Box\downarrow & \swarrow & \\
0 & \longrightarrow & \Gamma(T^*M) & \xrightarrow[2\nabla_S]{} & \Gamma(\otimes^2_S T^*M) & \xrightarrow[B]{} & \Gamma(\otimes^4_{\mathcal{R}} T^*M) & \longrightarrow & \cdots
\end{array} \tag{3.51}$$

where $\Gamma(\otimes^4_{\mathcal{R}} T^*M) \subset \Gamma(\otimes^4 T^*M)$ is the subset of Riemann-like symmetric tensors, *i.e.* 4-tensors t such that $t_{ab(cd)} = t_{(ab)cd} = t_{abcd} - t_{cdab} = t_{[abc]d} = 0$, $(Bh)_{abcd} := \nabla_{(a}\nabla_{c)}h_{bd} - \nabla_{(b}\nabla_{c)}h_{ad} - \nabla_{(a}\nabla_{d)}h_{bc} + \nabla_{(b}\nabla_{d)}h_{ac}$ and $(\mathrm{tr}'\,t)_{ab} := t_{acb}{}^{c}$ is the trace with respect to the second and fourth indices. The solid arrows in the diagram commute, while dashed arrows are the homotopical operators that induce the vertical cochain maps, *i.e.* it holds

$$\Box = 2\operatorname{div}I\nabla_S\,, \quad \text{on } \Gamma(T^*M)\,, \tag{3.52a}$$

$$\Box = 2\nabla_S\operatorname{div}I + \mathrm{tr}'\,B\,, \quad \text{on } \Gamma(\otimes^2_S T^*M)\,. \tag{3.52b}$$

We denote with $HC^i(M)$ the i-th cohomology of the Calabi complex. It is immediate to realize that $HC^0(M) \cong H_1(\mathfrak{Sol}(M))$.

We need also to consider the formal adjoint Calabi complex. It is obtained by taking the formal adjoint of each arrow in complex (3.51). We get

$$\begin{array}{ccccccccc}
0 & \longleftarrow & \Gamma(T^*M) & \xleftarrow{-2\,\mathrm{div}} & \Gamma(\otimes^2_S T^*M) & \xleftarrow{B^*} & \Gamma(\otimes^4_{\mathcal{R}} T^*M) & \longleftarrow & \cdots \\
 & & \Box\Big\uparrow & \nearrow_{-I\nabla_S} & \Box\Big\uparrow & \nearrow_{\frac{1}{4}g\odot} & \Box\Big\uparrow & \nearrow & \\
0 & \longleftarrow & \Gamma(T^*M) & \xleftarrow[-2\,\mathrm{div}]{} & \Gamma(\otimes^2_S T^*M) & \xleftarrow[B^*]{} & \Gamma(\otimes^4_{\mathcal{R}} T^*M) & \longleftarrow & \cdots
\end{array} \tag{3.53}$$

where $\odot$ is the Kulkarni-Nomizu product, $(g\odot h)_{abcd} \coloneqq g_{ac}h_{bd} - g_{bc}h_{ad} - g_{ad}h_{bc} + g_{bd}h_{ac}$, while B^* is given by $(B^*t)_{ab} \coloneqq 4\nabla^{(c}\nabla^{d)}t_{acbd}$. We denote the i-th homology of the formal adjoint Calabi complex with $HC'_i(M)$.

As a consequence of the following Proposition 3.2.11, on a spacetime M of constant curvature one has $H_{-1}(\mathfrak{Sol}(M)) = \ker \mathrm{div}/\operatorname{Im} P = \ker \mathrm{div}/\operatorname{Im} B^* = HC'_1(M)$. In [Kha16] the homology groups $HC'_i(M)$ are shown to be isomorphic to the cohomology groups of the sheaf of Killing-Yano tensors[1] on (M, g). ∇

Proposition 3.2.11. *Suppose M is a physical spacetime, as per Definition 2.1.9, of constant curvature, then $\operatorname{Im} P = \operatorname{Im} B^*$, where P is the dynamical operator of Equation (2.10) while B^* is the differential in the formal adjoint Calabi complex (3.53).*

Proof. We can use the homotopical operators in the complex of Equation (3.53) writing P in a more useful form. Indeed, it holds the identity $\Box = 2I\nabla_S\mathrm{div} + \frac{1}{4}B^*(g\odot -)$ on $\Gamma(\otimes^2_S T^*M)$ and, thence, we get

$$Ph = -\frac{1}{4}B^*\left(g\odot Ih\right)\,, \quad \forall h \in \Gamma(\otimes^2_S T^*M)\,. \tag{3.54}$$

We start by proving the set inclusion $\operatorname{Im} P \subseteq \operatorname{Im} B^*$. Let $t \in \operatorname{Im} P$, then there exists $h \in \Gamma(\otimes^2_S T^*M)$ such that $Ph = t$. From Equation (3.54) it follows $t = B^*\tilde{h}$, where $\tilde{h} \coloneqq -\frac{1}{4}g\odot Ih$. Therefore, $t \in \operatorname{Im} B^*$.

Consider now the opposite inclusion. Let $t \in \operatorname{Im} B^*$. There exists $\tilde{h} \in \Gamma(\otimes^4_{\mathcal{R}} T^*M)$ such that $t = B^*\tilde{h}$. We need to find $h \in \Gamma(\otimes^2_S T^*M)$ such that $Ph = t = B^*\tilde{h}$. We are going to construct h with the help of a partition of unity $\{f_+, f_-\}$ with the property that $f_\pm = 1$ on a past/future compact domain. We set

$$h \coloneqq G_+B^*(f_+\tilde{h}) + G_-B^*(f_-\tilde{h})\,. \tag{3.55}$$

Observe that h fulfills the de Donder condition:

$$\mathrm{div}\, Ih = \sum_{\alpha=\pm} \mathrm{div}\, IG_\alpha B^*(f_\alpha\tilde{h}) = -\sum_{\alpha=\pm} G^\Box_\alpha\, \mathrm{div}\, B^*(f_\alpha\tilde{h}) = 0\,, \tag{3.56}$$

[1] Killing-Yano tensors are solutions on (M, g) of the Killing-Yano equation, $\nabla_{(a}t_{b_1)b_2\ldots b_n} = 0$, for $t \in \Gamma(\otimes^n T^*M)$. For the definition of sheaves and for the construction of the Killing-Yano sheaf we refer to [KS05; Kha17].

where in the second step Lemma 2.2.5 is used and the last identity holds since $\operatorname{div} B^* = 0$, *cf.* Equation (3.53). We just need to check that the section we built is a solution for our problem. Let us calculate

$$Ph = P'h = \sum_{\alpha=\pm} P' G_\alpha B^*(f_\alpha \tilde{h}) = \sum_{\alpha=\pm} B^*(f_\alpha \tilde{h}) = B^* \tilde{h} = t\,, \tag{3.57}$$

where in the first step we used Equation (3.56), in the third the fact that $G_\pm$ are retarded/advanced Green operators for P' and finally the fourth one follows from the-linearity of B^*. Thence, $\operatorname{Im} B^* \subseteq \operatorname{Im} P$. We conclude that the equality $\operatorname{Im} B^* = \operatorname{Im} P$ holds true. □

Example 3.2.12. We give here an example of a constant curvature, non simply connected, Ricci-flat, Lorentzian manifold M whose corresponding solution complex $\mathfrak{Sol}(M)$ has non trivial homology groups in both degrees ± 1. It provides a concrete example of a spacetime where all the higher structures that we introduced come into play.

Let $\mathbb{M}^4 := (\mathbb{R}^4, \eta)$ be the Minkowski spacetime whose line element reads explicitly $\mathrm{d}\, s^2 = -\left(\mathrm{d}\, x^0\right)^2 + \sum_{i=1}^3 \left(\mathrm{d}\, x^i\right)^2$. We introduce the equivalence relation: $x^a \sim x^a + 1$, for $a = 1, 2, 3$. By taking the quotient by this equivalence relation, we contruct the manifold $M := \mathbb{R}^4/\sim$, which is isomorphic to $\mathbb{R} \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. We endow M with the Lorentzian metric induced by η via the quotient map $p : \mathbb{R}^4 \to M$. Therefore, M is a constant curvature, Ricci-flat, globally hyperbolic Lorentzian manifold. Observe that the Cauchy surface of M is isomorphic to $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ and thus it is compact. Moreover, M is not simply connected since its fundamental group is $\pi_1(M) \cong \mathbb{Z}^3$, as a quick computation reveals. From Remark 3.2.10 we know that the following isomorphisms hold true:

$$H_1(\mathfrak{Sol}(M)) \cong HC^0(M)\,, \quad H_{-1}(\mathfrak{Sol}(M)) \cong HC_1'(M)\,. \tag{3.58}$$

Furthermore, we can consider the analogue of Calabi complex (3.51) with compact and timelike compact supports. We denote the corresponding cohomology groups by $HC_c^i(M)$ and $HC_{tc}^i(M)$. Since M has a compact Cauchy surface compactness and timelike compactness of closed subsets are equivalent conditions, as it is shown in [Bae14]. By exploiting this and the analyses in [Kha16; Kha17], we manage to write the chain of isomorphisms

$$HC_1'(M) \cong HC_c^1(M)^* = HC_{tc}^1(M)^* \cong HC^0(M)^*\,. \tag{3.59}$$

Finally, isomorphisms in Equations (3.59) and (3.58) yield

$$H_{-1}(\mathfrak{Sol}(M)) \cong H_1(\mathfrak{Sol}(M))^*\,. \tag{3.60}$$

Therefore, we can compute only one of the two homology groups, the second being isomorphic to its dual. The computation of $H_1(\mathfrak{Sol}(M))$ can be performed following the ideas in [Kha17] for the non-simply connected case. According to it, the homology group $H_1(\mathfrak{Sol}(M))$ is isomorphic to $\mathfrak{g}^\pi$, *i.e.* the π-invariant subspace of $\mathfrak{g}$, where $\mathfrak{g}$ is the

Lie algebra of the isometries group of the universal cover of M and π is the composite adjoint monodromy representation of $\pi_1(M)$. Per construction, M is covered by $\mathbb{M}^4$, hence, $\mathfrak{g} \cong \mathbb{R}^4 \rtimes \mathfrak{so}(1,3)$ is the Poincaré Lie algebra. The composite adjoint monodromy representation is given by the composition $\mathrm{Ad} \circ \rho : \mathbb{Z}^3 \to \mathrm{Aut}\,(\mathfrak{g})$, where

$$\begin{aligned} \rho : \mathbb{Z}^3 &\longrightarrow \mathbb{R}^4 \rtimes O(1,3) \\ \begin{pmatrix} a \\ b \\ c \end{pmatrix} &\longmapsto \rho\begin{pmatrix} a \\ b \\ c \end{pmatrix} \coloneqq \left(\begin{pmatrix} 0 \\ a \\ b \\ c \end{pmatrix}, \mathbb{1} \right), \end{aligned} \tag{3.61}$$

is the action by isometries of deck transformations on $\mathbb{M}^4$, and

$$\begin{aligned} \mathrm{Ad} : \mathbb{R}^4 \rtimes O(1,3) &\longrightarrow \mathrm{Aut}\,(\mathfrak{g}) \\ (\mathbf{v}, \Lambda) &\longmapsto \left(\mathrm{Ad}(\mathbf{v}, \Lambda) : (\mathbf{p}, L) \longmapsto (\Lambda \mathbf{p} - \Lambda L \Lambda^{-1} \mathbf{v}, \Lambda L \Lambda^{-1}) \right), \end{aligned} \tag{3.62}$$

is the adjoint representation of the Poincaré group. Therefore, we write

$$\mathfrak{g}^\pi = \left\{ (\mathbf{p}, L) \in \mathfrak{g} \mid \mathrm{Ad} \circ \rho\begin{pmatrix} a \\ b \\ c \end{pmatrix}(\mathbf{p}, L) = (\mathbf{p}, L)\,,\ \forall a, b, c \in \mathbb{Z} \right\} \tag{3.63}$$

and, after a straightforward calculation, we get

$$\mathfrak{g}^\pi = \{ (\mathbf{p}, 0) \in \mathfrak{g} \} \cong \mathbb{R}^4 \,. \tag{3.64}$$

We conclude that both the homology groups of the solution complex are non trivial since the isomorphisms $H_{\pm 1}(\mathfrak{Sol}(M)) \cong \mathbb{R}^4$ hold true. △

3.3 Classical observable complex and Poisson structures

We are going to show that the solution complex $\mathfrak{Sol}(M)$ comes naturally endowed with a shifted Poisson structure. This could be physically interpreted as the antibracket of the BV-BRST formalism. In order to be able to construct a quantum theory for linearized gravity, we will need an unshifted Poisson structure. We will conclude showing that the solution complex can be endowed with such a structure, by relying heavily on the hypothesis that M is globally hyperbolic.

All these structures need to be defined on a *complex of linear observables*. Since this has to be dually paired to the solution complex, we give the following definition.

Definition 3.3.1 (Complex of linear observables)**.** Let M be a physical spacetime, as per Definition 2.1.9, and let $\mathfrak{Sol}(M)$ be the linearized gravity solution complex on M. The *complex of linear observables* for $\mathfrak{Sol}(M)$ is defined as its smooth dual. We denote it by $\mathfrak{Obs}(M)$ and we set

$$\mathfrak{Obs}(M) \coloneqq \left(\overset{(-1)}{\Gamma_c(T^*M)} \xleftarrow{\mathrm{div}} \overset{(0)}{\Gamma_c(\otimes_S^2 T^*M)} \xleftarrow{P} \overset{(1)}{\Gamma_c(\otimes_S^2 T^*M)} \xleftarrow{-\nabla_S} \overset{(2)}{\Gamma_c(T^*M)} \right). \tag{3.65}$$

Lemma 3.3.2. *The complex of linear observables, $\mathfrak{Obs}(M)$, as per Definition 3.3.1 is dually paired to the solution complex $\mathfrak{Sol}(M)$.*

Proof. The proof follows the same lines of the analogous one for the complex $\mathfrak{C}(M)^*$ in Section 3.2. The evaluation chain map,

$$(-,-) : \mathfrak{Obs}(M) \otimes \mathfrak{Sol}(M) \longrightarrow \mathbb{R}\,, \tag{3.66}$$

is built with the integral pairings as per Equations (3.19).

Let us write the degree 0 of the tensor product complex $\mathfrak{Obs}(M) \otimes \mathfrak{Sol}(M)$:

$$\begin{aligned}(\mathfrak{Obs}(M) \otimes \mathfrak{Sol}(M))_0 = (\Gamma_c(T^*M) \otimes \Gamma(T^*M)) \oplus \big(\Gamma_c(\otimes^2_S T^*M) \otimes \Gamma(\otimes^2_S T^*M)\big) \oplus \\ \oplus \big(\Gamma_c(\otimes^2_S T^*M) \otimes \Gamma(\otimes^2_S T^*M)\big) \oplus (\Gamma_c(T^*M) \otimes \Gamma(T^*M))\,.\end{aligned}$$

Observe that the spaces paired in this degree are consistent with our pairing, which yields a well defined map from the degree 0 to the real numbers.

Finally, one has to prove that the differentials in (3.65) make the pairing into a chain map. This can be shown by a direct calculation, once the differential of the tensor product complex is computed explicitly. □

We give the following physical interpretation of the elements of each degree of $\mathfrak{Obs}(M)$. This is a direct consequence of how the evaluation chain map acts, pairing observables and fields:

- elements $\varepsilon \in \mathfrak{Obs}(M)_0 = \Gamma_c(\otimes^2_S T^*M)$ are linear observables for the gauge fields;
- elements $\eta \in \mathfrak{Obs}(M)_{-1} = \Gamma_c(T^*M)$ are linear observables for the ghost fields;
- elements $\alpha \in \mathfrak{Obs}(M)_1 = \Gamma_c(\otimes^2_S T^*M)$ and $\beta \in \mathfrak{Obs}(M)_2 = \Gamma_c(T^*M)$ are linear observables for the antifields $h^\ddagger$ and $\chi^\ddagger$, respectively.

Let us write the evaluations of these observables on fields of $\mathfrak{Sol}(M)$ by making explicit the pairings. For each gauge field $h \in \mathfrak{Sol}(M)_0 = \Gamma(\otimes^2_S T^*M)$, ghost field $\chi \in \mathfrak{Sol}(M)_1 = \Gamma(T^*M)$ and antifields $h^\ddagger \in \mathfrak{Sol}(M)_{-1} = \Gamma(\otimes^2_S T^*M)$ and $\chi^\ddagger \in \mathfrak{Sol}(M)_{-2} = \Gamma(T^*M)$, we have

$$(\varepsilon, h) = \int_M \varepsilon_{ab} h_{cd} g^{ac} g^{bd} \mu_g\,, \qquad (\eta,\chi) = \int_M \eta_a \chi_b g^{ab} \mu_g\,, \tag{3.67a}$$

$$(\alpha, h^\ddagger) = \int_M \alpha_{ab} h^\ddagger_{cd} g^{ac} g^{bd} \mu_g\,, \qquad (\beta,\chi^\ddagger) = \int_M \beta_a \chi^\ddagger_b g^{ab} \mu_g\,. \tag{3.67b}$$

We can now write the homology groups of the complex of linear observables.

- $H_{-1}(\mathfrak{Obs}(M)) = \Gamma_c(T^*M)/\operatorname{Im}_c(\operatorname{div})$. It contains linear observables that test those ghost fields that act trivially on gauge fields, *cf.* Proposition 3.3.5 and Remark 3.3.6;

- $H_0(\mathfrak{Obs}(M)) = \mathrm{Ker}_c\,(\mathrm{div})/\,\mathrm{Im}_c\,P = \mathcal{E}$. This is the usual space of gauge invariant on-shell classical observables for linearized gravity. We have already found it by following another path in Section 2.3, Equation (2.50);
- $H_1(\mathfrak{Obs}(M)) = \mathrm{Ker}_c\,P/\,\mathrm{Im}_c\,\nabla_S$ is the space of linear observables testing obstructions to solving the inhomogeneous linearized gravity equation $Ph = t$ with $t \in \mathrm{Ker}\,(\mathrm{div})$, see again Proposition 3.3.5 and Remark 3.3.6;
- $H_2(\mathfrak{Obs}(M)) = \mathrm{Ker}_c\,\nabla_S$. This homology is always trivial, as we show with the following lemma.

Lemma 3.3.3. *Let M be a physical spacetime, see Definition 2.1.9. The Killing operator on compactly supported sections $\nabla_S : \Gamma_c(T^*M) \to \Gamma_c(\otimes_S^2 T^*M)$ has trivial kernel.*

Proof. Let $\eta \in \Gamma_c(T^*M)$ be a compactly supported section in the kernel of ∇_S. Therefore, it holds

$$\nabla_a \eta_b + \nabla_b \eta_a = 0\,. \tag{3.68}$$

Let us take now the divergence of (3.68):

$$0 = \nabla^a(\nabla_a \eta_b + \nabla_b \eta_a) = \Box \eta_b + \nabla_b \nabla^a \eta_a + R^{q\ \,a}_{\ a\ \,b}\eta_q = \Box \eta_b + \nabla_b \nabla^a \eta_a\,, \tag{3.69}$$

where we used the commutation relations between covariant derivatives and the fact that g is Ricci-flat per hypothesis. By contracting the identity (3.68) with the metric, we find that $\nabla^a \eta_a = 0$. By inserting this identity in Equation (3.69) we get $\Box \eta_b = 0$. From standard results on wave operators on globally hyperbolic manifolds, see [BGP08], we know that the wave operator $\Box$ on compactly supported sections $\Gamma_c(T^*M)$ has trivial kernel. This yields $\eta = 0$ and thus the statement is proved. □

Remark 3.3.4. The zeroth homology of the observable complex coincides with the space of classical on-shell gauge invariant observables. This has to be confronted with the fact that the zeroth homology $H_0(\mathfrak{Sol}(M))$ is the vector space of on-shell gauge equivalence classes of linearized gravity fields. Recalling the content of Chapter 2, we can highlight that these spaces are dually paired, that is, each $[\varepsilon] \in H_0(\mathfrak{Obs}(M))$ identifies a linear functional on the space of on-shell gauge equivalence classes $H_0(\mathfrak{Sol}(M))$ via the integral pairing (3.19a). ▽

There exist duality relations also between the other homology groups of the linear observable and solution complexes. This duality is due to the integral pairings between sections and compactly supported sections, which descend to the quotient.

Proposition 3.3.5. *Let M be a physical spacetime and let $\mathfrak{Sol}(M)$ and $\mathfrak{Obs}(M)$ be the corresponding solution and linear observable complexes previously introduced. Then the following inclusions hold:*

i. $H_1(\mathfrak{Obs}(M)) \subseteq H_{-1}(\mathfrak{Sol}(M))^*$,

ii. $H_{-1}(\mathfrak{Obs}(M)) \subseteq H_1(\mathfrak{Sol}(M))^*$, *where we are here considering algebraic dual spaces.*

Proof. Let us start with the first inclusion. We need to prove that it is possible to associate to each element $[\alpha] \in H_1(\mathfrak{Obs}(M)) = \mathrm{Ker}_c\, P/\,\mathrm{Im}_c\, \nabla_S$ a linear functional $\phi_{[\alpha]} : H_{-1}(\mathfrak{Sol}(M)) = \mathrm{Ker}\,(\mathrm{div})/\,\mathrm{Im}\, P \to \mathbb{R}$. This can be done by means of the integral pairing (3.19a). Let us define

$$\phi_{[\alpha]} : \mathrm{Ker}\,(\mathrm{div})/\,\mathrm{Im}\, P \longrightarrow \mathbb{R}\,, \quad [h^\ddagger] \longmapsto \phi_{[\alpha]}([h^\ddagger]) \coloneqq (\alpha, h^\ddagger)\,, \tag{3.70}$$

where in the last formula we picked arbitrary representatives in the equivalence classes. We need to prove that the definition does not depend on the choice of representatives. Therefore, let $\alpha \sim \alpha' = \alpha + \nabla_S \beta$ and $h^\ddagger \sim h'^\ddagger = h^\ddagger + Ph$, for $\beta \in \Gamma_c(T^*M)$ and $h \in \Gamma(\otimes_S^2 T^*M)$. We have

$$\begin{aligned}(\alpha', h'^\ddagger) &= (\alpha + \nabla_S\beta, h^\ddagger + Ph) = (\alpha, h^\ddagger) + (\alpha, Ph) + (\nabla_S\beta, h^\ddagger) + (\nabla_S\beta, Ph)\\ &= (\alpha, h^\ddagger) + (P\alpha, h) + (\beta, -\nabla_S h^\ddagger) + (P\nabla_S\beta, h) = (\alpha, h^\ddagger)\,,\end{aligned} \tag{3.71}$$

using Proposition 2.1.14 and the fact that $\alpha \in \mathrm{Ker}_c\, P$ and $h^\ddagger \in \mathrm{Ker}\,(\mathrm{div})$. The map $\phi_{[\alpha]}$ is thus well defined on the quotient and it is linear because the pairing is a bilinear map.

The second inclusion is proved in a similar fashion. Let $[\eta] \in H_{-1}(\mathfrak{Obs}(M)) = \Gamma_c(T^*M)/\,\mathrm{Im}_c\,(\mathrm{div})$ and define

$$\psi_{[\eta]} : H_1(\mathfrak{Sol}(M)) = \mathrm{Ker}\, \nabla_S \longrightarrow \mathbb{R}\,, \quad \chi \longmapsto \psi_{[\eta]}(\chi) \coloneqq (\eta, \chi)\,. \tag{3.72}$$

The pairing that appears in this definition is that in Equation (3.19b). Again, the definition is given in terms of an arbitrary representative in the equivalence class. We need to show that the definition is well-posed. Then, let $\eta \sim \eta'$ in $\Gamma_c(T^*M)/\,\mathrm{Im}_c\,(\mathrm{div})$. It means that there exist $\varepsilon \in \Gamma_c(\otimes_S^2 T^*M)$ such that $\eta' = \eta + \mathrm{div}\, \varepsilon$. Let us explicitly compute:

$$\psi_{[\eta]}(\chi) = (\eta', \chi) = (\eta + \mathrm{div}\,\varepsilon, \chi) = (\eta, \chi) + (\eta, -\nabla_S\chi) = (\eta, \chi)\,, \tag{3.73}$$

where we used Lemma 2.2.4 and the fact that $\chi \in \mathrm{Ker}\, \nabla_S$, per hypothesis. Therefore, the map $\psi_{[\eta]}$ is well defined on the quotient and it belongs to $H_1(\mathfrak{Sol}(M))^*$. □

Remark 3.3.6. Consider the special case of a background (M, g) of constant curvature. The homology groups of the complex of linear observables can be related to the cohomology counterparts of the compactly supported Calabi complex and of its formal adjoint complex. These are the subcomplexes of the complexes (3.51) and (3.53) with compactly supported sections. We introduce the following notation for the above mentioned (co)homology groups: $HC_c^i(M)$ is the i-th cohomology group of the Calabi complex with compact supports, while $HC_i(M)$ is the i-th homology group of the formal adjoint Calabi complex with compact supports. Observe that $HC_0(M)$ is exactly the same as $H_{-1}(\mathfrak{Obs}(M))$.

The following Proposition 3.3.7 reveals the connection between the homology groups of the complex of linear observables on constant curvature manifolds and the cohomology counterparts of Calabi complex with compact supports. Indeed, $H_1(\mathfrak{Obs}(M)) = \mathrm{Ker}_c\, P/\,\mathrm{Im}_c\, \nabla_S = \mathrm{Ker}_c\, B/\,\mathrm{Im}_c\, \nabla_S = HC^1_c(M)$.

For the case of constant curvature background manifolds, the identification of the homology groups of solution and of the linear observable complexes with suitable cohomology groups of the Calabi complexes allows us to state a stronger result than Proposition 3.3.5. Indeed, generalized Poincaré duality isomorphisms hold true between Calabi complex and formal adjoint Calabi complex with compact supports and between formal adjoint Calabi complex and Calabi complex with compact supports:

$$HC_i(M) \cong HC^i(M)^*, \qquad HC^i_c(M) \cong HC'_i(M)^*. \tag{3.74}$$

This result is proved in [Kha17, Corollary 11]. By making explicit the identities we derived in Remark 3.2.10 and Remark 3.3.6, the isomorphisms (3.74) yield

$$H_1(\mathfrak{Obs}(M)) = HC^1_c(M) \cong HC'_1(M)^* = H_{-1}(\mathfrak{Sol}(M))^*, \tag{3.75a}$$
$$H_{-1}(\mathfrak{Obs}(M)) = HC_0(M) \cong HC^0(M)^* = H_1(\mathfrak{Sol}(M))^*. \tag{3.75b}$$

$\triangledown$

Proposition 3.3.7. *Suppose M is a physical spacetime of constant curvature, see Definition 2.1.9. Then $\mathrm{Ker}_c\, P = \mathrm{Ker}_c\, B$, where P is the linearized gravity dynamical operator of Equation (2.10) while B is a differential in the Calabi complex (3.51).*

Proof. The homotopical operators in the Calabi complex (3.51) yield $\Box = 2\nabla_S \mathrm{div}\, I + \mathrm{tr}'\, B$ and thence $Ph = -I\, \mathrm{tr}'\,(Bh)$ on $\Gamma(\otimes^2_S T^*M)$. Let $h \in \mathrm{Ker}_c\, B$. It follows $Ph = 0$, that is $h \in \mathrm{Ker}_c\, P$. We conclude $\mathrm{Ker}_c\, B \subseteq \mathrm{Ker}_c\, P$. Vice versa let $h \in \mathrm{Ker}_c\, P$. By using the definition of P', $P = P' + 2I\nabla_S \mathrm{div}\, I$, we find

$$P'h = -2I\nabla_S \mathrm{div}\, Ih\,. \tag{3.76}$$

We recall that the differential operator P' is Green hyperbolic with retarded/advanced Green operators $G_\pm$. Hence,

$$h = G_+P'h = -2G_+I\nabla_S \mathrm{div}\, Ih = 2\nabla_S G^{\Box}_+ \,\mathrm{div}\, Ih\,, \tag{3.77}$$

where in the second step the identity (3.76) is inserted and in the last one it is used the dual of Lemma 2.2.5. We recall $G^{\Box}_+$ is the retarded Green operator for the wave operator $\Box = \nabla^a\nabla_a$ on $\Gamma(T^*M)$. Finally, consider that it holds $Bh = 2B\nabla_S G^{\Box}_+ \,\mathrm{div}\, Ih = 0$ since ∇_S and B are adjacent differentials in the Calabi complex (3.51). This shows that $h \in \mathrm{Ker}_c\, B$ and thence $\mathrm{Ker}_c\, P \subseteq \mathrm{Ker}_c\, B$. The two opposite set inclusions imply the sought equality, $\mathrm{Ker}_c\, P = \mathrm{Ker}_c\, B$. $\Box$

We want now to show that the observable complex comes endowed with a shifted Poisson structure. In order to do this, we first need to consider the 1-shifting of the complex $\mathfrak{Sol}(M)$ itself.

We recall that, for $(V, \mathrm{d}^V) \in \mathsf{Ch}_{\mathbb{R}}$ and $p \in \mathbb{Z}$, the p-shifting of (V, d^V) is the complex $(V[p], \mathrm{d}^{V[p]})$, as per Definition 1.1.20, which reads explicitly

$$\begin{aligned} V[p]_n &:= V_{n-p}\,, \\ \mathrm{d}_n^{V[p]} &:= (-1)^p\, \mathrm{d}_{n-p}^V\,. \end{aligned} \tag{3.78}$$

The complex we get by shifting the solution one reads

$$\mathfrak{Sol}(M)[1] = \left(\overset{(-1)}{\Gamma(T^*M)} \xleftarrow{\mathrm{div}} \overset{(0)}{\Gamma(\otimes_S^2 T^*M)} \xleftarrow{-P} \overset{(1)}{\Gamma(\otimes_S^2 T^*M)} \xleftarrow{-\nabla_S} \overset{(2)}{\Gamma(T^*M)} \right). \tag{3.79}$$

Then, let us consider the inclusion map $\iota : \Gamma_c(F) \to \Gamma(F)$, which embeds compactly supported sections in arbitrary sections, where F can be either T^*M or $\otimes_S^2 T^*M$. These maps allow us to define a chain map $j : \mathfrak{Obs}(M) \to \mathfrak{Sol}(M)[1]$

$$\begin{array}{c} \mathfrak{Obs}(M) \\ {\scriptstyle j}\big\downarrow \\ \mathfrak{Sol}(M)[1] \end{array} := \left(\begin{array}{ccccccc} \Gamma_c(T^*M) & \xleftarrow{\mathrm{div}} & \Gamma_c(\otimes_S^2 T^*M) & \xleftarrow{P} & \Gamma_c(\otimes_S^2 T^*M) & \xleftarrow{-\nabla_S} & \Gamma_c(T^*M) \\ {\scriptstyle \iota}\big\downarrow & & {\scriptstyle \iota}\big\downarrow & & {\scriptstyle -\iota}\big\downarrow & & {\scriptstyle -\iota}\big\downarrow \\ \Gamma(T^*M) & \xleftarrow[\mathrm{div}]{} & \Gamma(\otimes_S^2 T^*M) & \xleftarrow[-P]{} & \Gamma(\otimes_S^2 T^*M) & \xleftarrow[-\nabla_S]{} & \Gamma(T^*M) \end{array} \right) \tag{3.80}$$

We now define the shifted Poisson structure on $\mathfrak{Sol}(M)$.

Definition 3.3.8 (Shifted Poisson structure)**.** The *shifted Poisson structure* on the solution complex $\mathfrak{Sol}(M)$ is the chain map $\Upsilon : \mathfrak{Obs}(M) \otimes \mathfrak{Obs}(M) \to \mathbb{R}[1]$ given by the composition

$$\begin{array}{ccc} \mathfrak{Obs}(M) \otimes \mathfrak{Obs}(M) & \xrightarrow{\qquad\qquad \Upsilon \qquad\qquad} & \mathbb{R}[1] \\ {\scriptstyle \mathrm{id}\otimes j}\big\downarrow & & \big\uparrow {\scriptstyle \mathrm{id}\otimes(-,-)} \\ \mathfrak{Obs}(M) \otimes \mathfrak{Sol}(M)[1] & & \\ \wr\wr & & \\ \mathfrak{Obs}(M) \otimes \mathbb{R}[1] \otimes \mathfrak{Sol}(M) & \xrightarrow[\gamma\otimes\mathrm{id}]{} & \mathbb{R}[1] \otimes \mathfrak{Obs}(M) \otimes \mathfrak{Sol}(M) \end{array} \tag{3.81}$$

where γ is the symmetric braiding in the monoidal category $\mathsf{Ch}_{\mathbb{R}}$, $\gamma : V \otimes W \to W \otimes V$, which is given by Equation (1.14).

Remark 3.3.9. In the definition we used the isomorphism $\mathfrak{Sol}(M)[1] \cong \mathbb{R}[1] \otimes \mathfrak{Sol}(M)$. This follows directly from Definition 1.1.18 of tensor product of chain complexes. Indeed, for $n \in \mathbb{Z}$, we have

$$(\mathbb{R}[1] \otimes \mathfrak{Sol}(M))_n = \mathbb{R} \otimes \mathfrak{Sol}(M)_{n-1} \cong \mathfrak{Sol}(M)_{n-1} = \mathfrak{Sol}(M)[1]_n\,. \tag{3.82}$$

Furthermore, the graded Leibniz rule entails that the differentials agree. ▽

Proposition 3.3.10. *Let $\mathfrak{Obs}(M)$ be the complex of linear observables for linearized gravity on a physical spacetime M see Definition 2.1.9 and let $\Upsilon : \mathfrak{Obs}(M) \otimes \mathfrak{Obs}(M) \to$*

$\mathbb{R}[1]$ *be the shifted Poisson structure from Definition 3.3.8. Then, the Poisson structure reads explicitly*

$$\Upsilon(\varepsilon,\alpha) = -\int_M \varepsilon_{ab}\alpha_{cd}g^{ac}g^{bd}\mu_g = \Upsilon(\alpha,\varepsilon)\,, \tag{3.83a}$$

$$\Upsilon(\eta,\beta) = \int_M \eta_a\beta_b g^{ab}\mu_g = \Upsilon(\beta,\eta)\,, \tag{3.83b}$$

for all $\varepsilon \in \mathfrak{Obs}(M)_0 = \Gamma_c(\otimes^2_S T^*M)$, $\alpha \in \mathfrak{Obs}(M)_1 = \Gamma_c(\otimes^2_S T^*M)$, $\eta \in \mathfrak{Obs}(M)_{-1} = \Gamma_c(T^*M)$ *and* $\beta \in \mathfrak{Obs}(M)_2 = \Gamma_c(T^*M)$.

Proof. We have to unravel Definition 3.3.8, keeping track of all the signs. Let us compute the degree 1 of the tensor product $\mathfrak{Obs}(M)\otimes\mathfrak{Obs}(M)$. It is sufficient to consider that since the target complex of the chain map Υ is concentrated in degree 1. We have

$$\begin{aligned}(\mathfrak{Obs}(M)\otimes\mathfrak{Obs}(M))_1 = \Big(\Gamma_c(\overset{(-1)}{T^*M})\otimes\Gamma_c(\overset{(2)}{T^*M})\Big) \oplus \Big(\Gamma_c(\overset{(0)}{\otimes^2_S T^*M})\otimes\Gamma_c(\overset{(1)}{\otimes^2_S T^*M})\Big)\\ \oplus \Big(\Gamma_c(\overset{(1)}{\otimes^2_S T^*M})\otimes\Gamma_c(\overset{(0)}{\otimes^2_S T^*M})\Big) \oplus \Big(\Gamma_c(\overset{(2)}{T^*M})\otimes\Gamma_c(\overset{(-1)}{T^*M})\Big).\end{aligned}$$

In Equation (3.81), the chain map j acts on the second tensor factor, changing the sign of elements in degrees 1 and 2. The second map sees the symmetric braiding acting on $\mathfrak{Obs}(M)\otimes\mathbb{R}[1]$. Since the factor $\mathbb{R}[1]$ is concentrated in degree 1, γ attaches a minus sign to elements in the odd degrees of $\mathfrak{Obs}(M)$. The combined action of these two maps changes the sign of the tensor product of symmetric 2-tensor sections, leaving unchanged the one of covector fields. Indeed,

- $\Gamma_c(\overset{(-1)}{T^*M})\otimes\Gamma_c(\overset{(2)}{T^*M}) \ni \eta\otimes\beta \xmapsto{\mathrm{id}\otimes j} \eta\otimes(-\beta) \xmapsto{\gamma\otimes\mathrm{id}} (-\eta)\otimes(-\beta)\,;$
- $\Gamma_c(\overset{(0)}{\otimes^2_S T^*M})\otimes\Gamma_c(\overset{(1)}{\otimes^2_S T^*M}) \ni \varepsilon\otimes\alpha \xmapsto{\mathrm{id}\otimes j} \varepsilon\otimes(-\alpha) \xmapsto{\gamma\otimes\mathrm{id}} \varepsilon\otimes(-\alpha)\,;$
- $\Gamma_c(\overset{(1)}{\otimes^2_S T^*M})\otimes\Gamma_c(\overset{(0)}{\otimes^2_S T^*M}) \ni \alpha\otimes\varepsilon \xmapsto{\mathrm{id}\otimes j} \alpha\otimes\varepsilon \xmapsto{\gamma\otimes\mathrm{id}} (-\alpha)\otimes\varepsilon\,;$
- $\Gamma_c(\overset{(2)}{T^*M})\otimes\Gamma_c(\overset{(-1)}{T^*M}) \ni \beta\otimes\eta \xmapsto{\mathrm{id}\otimes j} \beta\otimes\eta \xmapsto{\gamma\otimes\mathrm{id}} \beta\otimes\eta\,.$

By applying the section pairing and thanks to its symmetry, the equations (3.83) of the statement descend. □

Remark 3.3.11. Observe that the Poisson structure from Proposition 3.3.10 can be interpreted physically as a pairing between observables for gauge fields and its antifields, see Equation (3.83a), and between observables for ghost fields and its antifields, see Equation (3.83b). ▽

Our next aim is to construct the differential graded counterpart of the usual Poisson structure. Thence, we need to unshift the structure Υ in order to get a pairing whose target are real numbers concentrated in degree 0, instead of degree 1. In particular, we expect this unshifted structure also to pair together gauge field observables, just like the Poisson structure in Equation (2.53) does.

This construction relies on the introduction of some new maps we shall call retarded/advanced trivializations. This name is rather evocative and, in fact, these maps play a role very similar to that of retarded/advanced Green operators in ordinary field theories.

First, let us introduce the following complex: It is the past/future compact analogue of the complex of linear observables of Equation (3.65), *i.e.*

$$\mathfrak{Obs}_{\mathrm{pc/fc}}(M) := \left(\overset{(-1)}{\Gamma_{pc/fc}(T^*M)} \xleftarrow{\mathrm{div}} \overset{(0)}{\Gamma_{pc/fc}(\otimes_S^2 T^*M)} \xleftarrow{P} \overset{(1)}{\Gamma_{pc/fc}(\otimes_S^2 T^*M)} \xleftarrow{-\nabla_S} \overset{(2)}{\Gamma_{pc/fc}(T^*M)} \right). \tag{3.84}$$

Consider now the chain map j defined in Equation (3.80). This factors through the inclusion chain map $\iota : \mathfrak{Obs}(M) \to \mathfrak{Obs}_{\mathrm{pc/fc}}(M)$ which is simply given by the inclusion of compact sections in past/future compact ones. Therefore, we have the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{Obs}(M) & \xrightarrow{\quad j \quad} & \mathfrak{Sol}(M)[1] \\ & {}_{\iota}\searrow \quad \nearrow_{j_{\mathrm{pc/fc}}} & \\ & \mathfrak{Obs}_{\mathrm{pc/fc}}(M) & \end{array} \tag{3.85}$$

The map $j_{\mathrm{pc/fc}} : \mathfrak{Obs}_{\mathrm{pc/fc}}(M) \to \mathfrak{Sol}(M)[1]$ which appears in the triangle above is the extension of the chain map (3.80) to sections with past/future compact support.

Let us now define the notion of retarded/advanced trivializations.

Definition 3.3.12 (Retarded/advanced trivialization)**.** Let M be a physical spacetime and let $\mathfrak{Obs}_{\mathrm{pc/fc}}(M)$ be the complex of past/future compact linear observables defined in Equation (3.84). A *retarded/advanced trivialization* is a contracting homotopy of $\mathfrak{Obs}_{\mathrm{pc/fc}}(M)$, *i.e.* it is a 1-chain $\Lambda^{\pm} \in \underline{\mathrm{hom}}\,(\mathfrak{Obs}_{\mathrm{pc/fc}}(M), \mathfrak{Obs}_{\mathrm{pc/fc}}(M))_1$ such that $\partial\Lambda^{\pm} = \mathrm{id}$. We recall that the mapping complex $(\underline{\mathrm{hom}}, \partial)$ has been introduced in Definition 1.1.10.

Remark 3.3.13. Explicitly, a retarded/advanced trivialization is a family of linear homomorphisms $\Lambda^{\pm}_n : \mathfrak{Obs}_{\mathrm{pc/fc}}(M)_n \to \mathfrak{Obs}_{\mathrm{pc/fc}}(M)_{n+1}$ for $n \in \mathbb{Z}$, such that it holds $\mathrm{d}_{n+1}\,\Lambda^{\pm}_n + \Lambda^{\pm}_{n-1}\,\mathrm{d}_n = \mathrm{id}_n$ for all $n \in \mathbb{Z}$.

We can visualize a retarded/advanced trivialization by the dashed down-right pointing arrows in the diagram below.

$$
\begin{array}{ccccccccccc}
0 & \longleftarrow & \Gamma_{pc/fc}(T^*M) & \xleftarrow{\text{div}} & \Gamma_{pc/fc}(\otimes_S^2 T^*M) & \xleftarrow{P} & \Gamma_{pc/fc}(\otimes_S^2 T^*M) & \xleftarrow{-\nabla_S} & \Gamma_{pc/fc}(T^*M) & \longleftarrow & 0 \\
\downarrow & \searrow & \downarrow \text{id} & \overset{\Lambda^\pm_{-1}}{\searrow} & \downarrow \text{id} & \overset{\Lambda^\pm_0}{\searrow} & \downarrow \text{id} & \overset{\Lambda^\pm_1}{\searrow} & \downarrow \text{id} & \searrow & \downarrow \\
0 & \longleftarrow & \Gamma_{pc/fc}(T^*M) & \xleftarrow[\text{div}]{} & \Gamma_{pc/fc}(\otimes_S^2 T^*M) & \xleftarrow[P]{} & \Gamma_{pc/fc}(\otimes_S^2 T^*M) & \xleftarrow[-\nabla_S]{} & \Gamma_{pc/fc}(T^*M) & \longleftarrow & 0
\end{array}
\tag{3.86}
$$

From this diagram we infer that there are only three non-trivial components in a retarded/advanced trivialization, namely $\Lambda^\pm_{-1} : \Gamma_{pc/fc}(T^*M) \to \Gamma_{pc/fc}(\otimes_S^2 T^*M)$, $\Lambda^\pm_0 : \Gamma_{pc/fc}(\otimes_S^2 T^*M) \to \Gamma_{pc/fc}(\otimes_S^2 T^*M)$ and $\Lambda^\pm_1 : \Gamma_{pc/fc}(\otimes_S^2 T^*M) \to \Gamma_{pc/fc}(T^*M)$. ▽

At this point, it is necessary to address the problem of existence and uniqueness of retarded/advanced trivializations for the past/future compact linear observable complex. Let us start with existence by presenting a concrete realization of them.

Proposition 3.3.14. *Let M be a physical spacetime as per Definition 2.1.9. Denote by $G_\pm : \Gamma_{pc/fc}(\otimes_S^2 T^*M) \to \Gamma_{pc/fc}(\otimes_S^2 T^*M)$ the extended retarded/advanced Green operators for the differential operator $P' = (-\Box + 2\,\mathrm{Riem})I$ we introduced in Equation (3.48). Then, the maps*

$$
\Lambda^\pm_{-1} := -2G_\pm \nabla_S\,, \quad \Lambda^\pm_0 := G_\pm\,, \quad \Lambda^\pm_1 := 2\,\mathrm{div}\, G_\pm\,, \tag{3.87}
$$

define a retarded/advanced trivialization for linearized gravity.

Proof. Observe that the choices in Equation (3.87) are consistent with the domains and target spaces for the components of the 1-chain $\Lambda^\pm$. In order to show that they identify a contracting homotopy for $\mathfrak{Obs}_{\mathrm{pc/fc}}(M)$, it is enough to check that the identity $\partial\Lambda^\pm = \mathrm{id}$ holds. Referring to diagram (3.86), one has to verify that

$$
\mathrm{div} \circ \Lambda^\pm_{-1} = \mathrm{id}_{\Gamma_{pc/fc}(T^*M)}\,; \tag{3.88a}
$$
$$
\Lambda^\pm_{-1} \circ \mathrm{div} + P \circ \Lambda^\pm_0 = \mathrm{id}_{\Gamma_{pc/fc}(\otimes_S^2 T^*M)}\,; \tag{3.88b}
$$
$$
\Lambda^\pm_0 \circ P - \nabla_S \circ \Lambda^\pm_1 = \mathrm{id}_{\Gamma_{pc/fc}(\otimes_S^2 T^*M)}\,; \tag{3.88c}
$$
$$
-\Lambda^\pm_1 \circ \nabla_S = \mathrm{id}_{\Gamma_{pc/fc}(T^*M)}\,. \tag{3.88d}
$$

With $\Lambda^\pm_{-1} = -2G_\pm\nabla_S$ the identity (3.88a) is a by-product of Lemma 3.2.9.

Let us check the second identity by inserting $\Lambda^\pm_{-1} = -2G_\pm\nabla_S$, $\Lambda^\pm_0 = G_\pm$ and the identity $P = P' + 2I\nabla_S \mathrm{div}\, I$:

$$
\begin{aligned}
-2G_\pm\nabla_S\mathrm{div} + (P' + 2I\nabla_S\mathrm{div} I)G_\pm &= -2G_\pm\nabla_S\mathrm{div} + \mathrm{id} + 2I\nabla_S\mathrm{div}\, IG_\pm \\
&= -2G_\pm\nabla_S\mathrm{div} + \mathrm{id} - 2I\nabla_S G^\Box_\pm\,\mathrm{div} \\
&= -2G_\pm\nabla_S\mathrm{div} + \mathrm{id} + 2IG_\pm I\nabla_S\mathrm{div} = \mathrm{id}\,,
\end{aligned} \tag{3.89}
$$

where in the first step we used $P' \circ G_\pm = \mathrm{id}$, while in the second and in third ones Lemma 2.2.5 and its dual are respectively used. As far as the identity (3.88c) is concerned, we write

$$\begin{aligned} G_\pm P - 2\nabla_S \mathrm{div}\, G_\pm &= \mathrm{id} + 2G_\pm I \nabla_S \mathrm{div}\, I - 2\nabla_S \mathrm{div}\, G_\pm \\ &= \mathrm{id} - 2\nabla_S G_\pm^\square \,\mathrm{div}\, I - 2\nabla_S \mathrm{div}\, G_\pm \\ &= \mathrm{id} + 2\nabla_S \mathrm{div}\, I G_\pm I - 2\nabla_S \mathrm{div}\, G_\pm = \mathrm{id}\,, \end{aligned} \tag{3.90}$$

where again the identity $G_\pm \circ P' = \mathrm{id}$, Lemma 2.2.5 and its dual are used.

Finally, it remains to be proved the identity (3.88d). Let us use $\Lambda_1^\pm = 2\,\mathrm{div}\, G_\pm$ and let us compute

$$\begin{aligned} -2\,\mathrm{div}\, G_\pm \nabla_S = -2\,\mathrm{div}\, I^2 G_\pm \nabla_S &= -2\,\mathrm{div}\, I G_\pm I \nabla_S \\ &= 2G_\pm^\square \,\mathrm{div}\, I \nabla_S = G_\pm^\square \square = \mathrm{id}\,, \end{aligned} \tag{3.91}$$

where we used Proposition 2.1.11, Lemma 2.2.5 and the last identity in Lemma 2.1.13. □

Let us consider the question of uniqueness of retarded/advanced trivializations. We are going to show that those trivializations are unique up to homotopy. In the case of ordinary field theory, *e.g.* for Klein-Gordon theroy, this corresponds to the usual uniqueness of the retarded/advanced Green operator.

Lemma 3.3.15. *In the same setting of the Definition 3.3.12, let $\Lambda^\pm, \widetilde{\Lambda}^\pm$ be two retarded/advanced trivializations for linearized gravity, then there exists a 2-chain $\lambda^\pm \in \underline{\mathrm{hom}}\,(\mathfrak{Obs}_{\mathrm{pc/fc}}(M), \mathfrak{Obs}_{\mathrm{pc/fc}}(M))_2$ such that $\widetilde{\Lambda}^\pm - \Lambda^\pm = \partial\lambda^\pm$.*

Proof. First, observe that all the homology groups of the complex $\mathfrak{Obs}_{\mathrm{pc/fc}}(M)$ are trivial. This is an immediate consequence of the existence of a contracting homotopy $\Lambda^\pm$ for that complex. See as a reference [Wei95]. In other words, $\mathfrak{Obs}_{\mathrm{pc/fc}}(M)$ is quasi-isomorphic to 0. Because all objects in the model category $\mathsf{Ch}_\mathbb{R}$ are both fibrant and cofibrant the mapping complex functor $\underline{\mathrm{hom}}$ preserves quasi-isomorphisms (Lemma 1.1.19). Thence, the homology of $\underline{\mathrm{hom}}\,(\mathfrak{Obs}_{\mathrm{pc/fc}}(M), \mathfrak{Obs}_{\mathrm{pc/fc}}(M))$ is trivial too. Let $\Lambda^\pm, \widetilde{\Lambda}^\pm$ be two retarded/advanced trivializations, then their difference $\widetilde{\Lambda}^\pm - \Lambda^\pm$ is closed: $\partial(\widetilde{\Lambda}^\pm - \Lambda^\pm) = \mathrm{id} - \mathrm{id} = 0$. Since $\underline{\mathrm{hom}}\,(\mathfrak{Obs}_{\mathrm{pc/fc}}(M), \mathfrak{Obs}_{\mathrm{pc/fc}}(M))$ has trivial homology groups, it must be exact. Hence there exists $\lambda^\pm \in \underline{\mathrm{hom}}\,(\mathfrak{Obs}_{\mathrm{pc/fc}}(M), \mathfrak{Obs}_{\mathrm{pc/fc}}(M))_2$ such that $\widetilde{\Lambda}^\pm - \Lambda^\pm = \partial\lambda^\pm$. □

Corollary 3.3.16. *The retarded/advanced trivializations for linearized gravity are parameterized by 2-chains $\lambda^\pm \in \underline{\mathrm{hom}}\,(\mathfrak{Obs}_{\mathrm{pc/fc}}(M), \mathfrak{Obs}_{\mathrm{pc/fc}}(M))_2$. In other words, all the retarded/advanced trivializations for linearized gravity on a physical spacetime M, see Definition 2.1.9 are of the form*

$$\widetilde{\Lambda}_{-1}^\pm = -2G_\pm \nabla_S + P\lambda_{-1}^\pm\,, \tag{3.92a}$$

$$\widetilde{\Lambda}_0^\pm = G_\pm - \nabla_S \lambda_0^\pm - \lambda_{-1}^\pm \,\mathrm{div}\,, \tag{3.92b}$$

$$\widetilde{\Lambda}_1^\pm = 2\,\mathrm{div}\, G_\pm - \lambda_0^\pm P\,, \tag{3.92c}$$

for some linear maps $\lambda_0^\pm : \Gamma_{pc/fc}(\otimes_S^2 T^*M) \to \Gamma_{pc/fc}(T^*M)$ *and* $\lambda_{-1}^\pm : \Gamma_{pc/fc}(T^*M) \to \Gamma_{pc/fc}(\otimes_S^2 T^*M)$.

Proof. This is a consequence of Lemma 3.3.15 and Proposition 3.3.14.

Indeed, Proposition 3.3.14 provides us with a retarded/advanced trivialization, namely $\Lambda^\pm = (-2G_\pm \nabla_S, G_\pm, 2\,\mathrm{div}\, G_\pm)$. Then, from Lemma 3.3.15 we learn that for any other trivialization $\widetilde{\Lambda}^\pm$ it holds $\widetilde{\Lambda}^\pm = \Lambda^\pm + \partial\lambda^\pm$, for some 2-chain $\lambda^\pm$. The formulae (3.92) are obtained by making explicit the action of the boundary ∂ on 2-chains. Observe that the only non-zero components of a 2-chain $\lambda^\pm$ are the ones starting from the degrees -1 and 0. □

We have proved that, with general hypotheses on the background manifold M, the past/future compact linear observable complex admits retarded/advanced trivializations and they are unique up to homotopy. This result generalizes the one concerning Green operators for an ordinary field theory, but the non-uniqueness of the trivializations introduces some subtleties. In fact, retarded/advanced Green operators $G_\pm$ are such that $G_+^* = G_-$ and, thence, a formally skew-adjoint causal propagator $G = G_+ - G_-$ may be defined. If a similar construction is requested for retarded/advanced trivializations too, a criterion to select a pair of retarded and advanced trivializations is needed.

We start with a lemma ensuring that a chain map can be obtained from the difference of retarded and advanced trivializations.

Lemma 3.3.17. *Let* $\Lambda^\pm \in \underline{\mathrm{hom}}\left(\mathfrak{Obs}_{\mathrm{pc/fc}}(M), \mathfrak{Obs}_{\mathrm{pc/fc}}(M)\right)_1$ *be a pair of retarded and advanced trivializations. Then,*

i. it holds $j = \partial(j_{\mathrm{pc/fc}}\Lambda^\pm\iota)$ *and* $\Upsilon = \partial\big((\mathrm{id}\otimes(-,-))(\gamma\otimes\mathrm{id})(\mathrm{id}\otimes(j_{\mathrm{pc/fc}}\Lambda^\pm\iota))\big)$ *;*

ii. the difference

$$\Lambda \coloneqq j_{\mathrm{pc}}\Lambda^+\iota - j_{\mathrm{fc}}\Lambda^-\iota \in \underline{\mathrm{hom}}\left(\mathfrak{Obs}(M), \mathfrak{Sol}(M)[1]\right)_1 \tag{3.93}$$

is a closed 1*-chain, i.e.* $\partial\Lambda = 0$*. Moreover, it defines a chain map to the unshifted solution complex that we will denote with the same symbol,* $\Lambda : \mathfrak{Obs}(M) \to \mathfrak{Sol}(M)$.

Proof. Both proofs consist in straightforward checks. Let us start with item i.. Observe that $j_{\mathrm{pc/fc}}$ and ι are chain maps and thus they commute with the differentials of the complexes:

$$\mathrm{d}\, j_{\mathrm{pc/fc},n} = j_{\mathrm{pc/fc},n-1}\,\mathrm{d}\,, \quad \mathrm{d}\,\iota_n = \iota_{n-1}\,, \tag{3.94}$$

for each $n \in \mathbb{Z}$. Then,

$$\begin{aligned}\big(\partial(j_{\mathrm{pc/fc}}\Lambda^\pm\iota)\big)_n &= \mathrm{d}(j_{\mathrm{pc/fc}}\Lambda^\pm\iota)_n + (j_{\mathrm{pc/fc}}\Lambda^\pm\iota)_{n-1}\,\mathrm{d} \\ &= j_{\mathrm{pc/fc},n}\,\mathrm{d}\,\Lambda_n^\pm\iota_n + j_{\mathrm{pc/fc},n}\Lambda_{n-1}^\pm\,\mathrm{d}\,\iota_n \\ &= j_{\mathrm{pc/fc},n}(\partial\Lambda^\pm)_n\iota_n = j_{\mathrm{pc/fc},n}\iota_n = j_n\,,\end{aligned} \tag{3.95}$$

where the last steps follow from Definition 3.3.12 and from Equation (3.85). Second identity is proved in the same way. As an immediate consequence the homology classes of both j and Υ are trivial, namely $[j]=0$ and $[\Upsilon]=0$. Let us turn to the second part of the lemma. The fact that Λ is a 1-cycle is a direct consequence of what has just been proven. Indeed, $\partial\Lambda=\partial(j_{\mathrm{pc}}\Lambda^{+}\iota)-\partial(j_{\mathrm{fc}}\Lambda^{-}\iota)=j-j=0$. We conclude by observing that there is a chain isomorphism between $\underline{\mathrm{hom}}\,(V,W[p])$ and $\underline{\mathrm{hom}}\,(V,W)[p]$ for each pair of chain complexes V,W and $p\in\mathbb{Z}$. This isomorphism is concretely given by

$$\underline{\mathrm{hom}}\,(V,W[p])_n\ni(L_m:V_m\to W[p]_{m+n})_{m\in\mathbb{Z}}\longmapsto(L_m:V_m\to W_{m+n-p})_{m\in\mathbb{Z}}\in\underline{\mathrm{hom}}\,(V,W)_{n-p}\,. \tag{3.96}$$

Via this isomorphism, we can identify the 1-cycle Λ with a 0-cycle, *i.e.* a chain map, to $\mathfrak{Sol}(M)$, which we still denote with Λ. □

We are now ready to give a notion of compatible trivializations that mimics the sought behavior of Green operators.

Definition 3.3.18. A pair $\Lambda^{\pm}\in\underline{\mathrm{hom}}\,(\mathfrak{Obs}_{\mathrm{pc/fc}}(M),\mathfrak{Obs}_{\mathrm{pc/fc}}(M))_1$ of retarded and advanced trivializations is called *compatible* if the corresponding chain map Λ introduced in Lemma 3.3.17, item ii., enjoys a formal skew-adjointness property with respect to the pairing (3.66):

$$\begin{array}{ccc}\mathfrak{Obs}(M)\otimes\mathfrak{Obs}(M) & \xrightarrow{\ \mathrm{id}\otimes\Lambda\ } & \mathfrak{Obs}(M)\otimes\mathfrak{Sol}(M)\\ {\scriptstyle -\Lambda\otimes\mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle (-,-)}\\ \mathfrak{Sol}(M)\otimes\mathfrak{Obs}(M) & \xrightarrow[\ (-,-)\circ\gamma\ ]{} & \mathbb{R}\end{array} \tag{3.97}$$

Lemma 3.3.19. *The retarded/advanced trivializations explicitly written in Proposition 3.3.14 yield a compatible pair.*

Proof. We recall the identities $G^*=-G$, $\mathrm{div}^*=-\nabla_S$, which follow from general properties of Green operators and from Lemma 2.2.4. In order to check the skew-adjointness property (3.97), we are interested only in degree 0:

$$\begin{aligned}(\mathfrak{Obs}(M)\otimes\mathfrak{Obs}(M))_0=&\Big(\Gamma_c(\overset{(-1)}{T^*M})\otimes\Gamma_c(\overset{(1)}{\otimes_S^2T^*M})\Big)\\ &\oplus\Big(\Gamma_c(\overset{(0)}{\otimes_S^2T^*M})\otimes\Gamma_c(\overset{(0)}{\otimes_S^2T^*M})\Big)\\ &\oplus\Big(\Gamma_c(\overset{(1)}{\otimes_S^2T^*M})\otimes\Gamma_c(\overset{(-1)}{T^*M})\Big)\,.\end{aligned} \tag{3.98}$$

Moreover, the chain map Λ has only three non vanishing components, namely $\Lambda_{-1}=\Lambda^{+}_{-1}-\Lambda^{-}_{-1}=-2G\nabla_S$, $\Lambda_0=-\left(\Lambda^{+}_0-\Lambda^{-}_0\right)=-G$ and $\Lambda_1=-\left(\Lambda^{+}_1-\Lambda^{-}_1\right)=-2\,\mathrm{div}\,G$. Here we dropped the inclusion maps ι since no confusion should arise.

Let $\varepsilon_1 \otimes \varepsilon_2 \in \Gamma_c(\otimes_S^2 T^*M) \otimes \Gamma_c(\otimes_S^2 T^*M)$ and $\eta \otimes \alpha \in \Gamma_c(T^*M) \otimes \Gamma_c(\otimes_S^2 T^*M)$. Then,

$$\varepsilon_1 \otimes \varepsilon_2 \xmapsto{\mathrm{id}\otimes\Lambda} \varepsilon_1 \otimes \Lambda_0 \varepsilon_2 = -\varepsilon_1 \otimes G\varepsilon_2 \xmapsto{(-,-)} (\varepsilon_1, -G\varepsilon_2) = (G\varepsilon_1, \varepsilon_2)$$
$$\varepsilon_1 \otimes \varepsilon_2 \xmapsto{-\Lambda\otimes\mathrm{id}} -\Lambda_0 \varepsilon_1 \otimes \varepsilon_2 = G\varepsilon_1 \otimes \varepsilon_2 \xmapsto{\gamma} \varepsilon_2 \otimes G\varepsilon_1 \xmapsto{(-,-)} (\varepsilon_2, G\varepsilon_1)$$

and

$$\eta \otimes \alpha \xmapsto{\mathrm{id}\otimes\Lambda} \eta \otimes \Lambda_1 \alpha = -2\eta \otimes \operatorname{div} G\alpha \xmapsto{(-,-)} (\eta, -2\operatorname{div} G\alpha) = (-2G\nabla_S \eta, \alpha)$$
$$\eta \otimes \alpha \xmapsto{-\Lambda\otimes\mathrm{id}} -\Lambda_{-1}\eta \otimes \alpha = 2G\nabla_S \eta \otimes \alpha \xmapsto{\gamma} \alpha \otimes (-2G\nabla_S \eta) \xmapsto{(-,-)} (\alpha, -2G\nabla_S \eta)$$

The identity on the last component follows from symmetry. □

We are ready to give the definition of an unshifted Poisson structure on the solution complex $\mathfrak{Sol}(M)$.

Definition 3.3.20 (Unshifted Poisson structure)**.** Let $\mathfrak{Obs}_{\mathrm{pc/fc}}(M)$ be the complex as per Equation (3.84) and let $\Lambda^\pm \in \underline{\mathrm{hom}}\left(\mathfrak{Obs}_{\mathrm{pc/fc}}(M), \mathfrak{Obs}_{\mathrm{pc/fc}}(M)\right)_1$ be a compatible pair of retarded/advanced trivializations and denote with Λ the corresponding chain map as per Lemma 3.3.17, item ii.. The *unshifted Poisson structure* on the solution complex $\mathfrak{Sol}(M)$ is the chain map $\tau : \mathfrak{Obs}(M) \otimes \mathfrak{Obs}(M) \to \mathbb{R}$ defined by the composition

$$\begin{array}{ccc} \mathfrak{Obs}(M) \otimes \mathfrak{Obs}(M) & \xrightarrow{\quad \frac{1}{2}\tau \quad} & \mathbb{R} \\ & {\scriptstyle \mathrm{id}\otimes\Lambda} \searrow \qquad \nearrow {\scriptstyle (-,-)} & \\ & \mathfrak{Obs}(M) \otimes \mathfrak{Sol}(M) & \end{array} \tag{3.99}$$

where $\mathfrak{Obs}(M)$ is the complex of linear observables of Definition 3.3.1.

Corollary 3.3.21. *Let $\mathfrak{Sol}(M)$ be the solution complex and let $\mathfrak{Obs}(M)$ be the complex of linear observables for linearized gravity on a physical spacetime M, see Definition 2.1.9. Let $\Lambda^\pm$ be the pair of compatible retarded/advanced trivializations of Proposition 3.3.14 and denote with $\tau : \mathfrak{Obs}(M) \otimes \mathfrak{Obs}(M) \to \mathbb{R}$ the corresponding unshifted Poisson structure. Explicitly,*

$$\tau(\varepsilon_1, \varepsilon_2) = -2 \int_M (\varepsilon_1)_{ab} (G\varepsilon_2)_{cd} g^{ac} g^{bd} \mu_g = -\tau(\varepsilon_2, \varepsilon_1)\,, \tag{3.100a}$$

$$\tau(\eta, \alpha) = -4 \int_M \eta_a (\operatorname{div} G\alpha)_b g^{ab} \mu_g = \tau(\alpha, \eta)\,, \tag{3.100b}$$

for all $\varepsilon_1, \varepsilon_2 \in \mathfrak{Obs}(M)_0 = \Gamma_c(\otimes_S^2 T^*M)$, $\eta \in \mathfrak{Obs}(M)_{-1} = \Gamma_c(T^*M)$ and $\alpha \in \mathfrak{Obs}(M)_1 = \Gamma_c(\otimes_S^2 T^*M)$, where G is the causal propagator for the dynamical operator P'.

Proof. The action of the unshifted Poisson structure has already been revealed while proving Lemma 3.3.19. Formulae (3.100) are then obtained by making explicit the integral pairings within. □

Remark 3.3.22. Observe that the unshifted Poisson structure from Corollary 3.3.21 can be interpreted physically as a pairing between two linear observables for gauge fields, see Equation (3.100a), and between linear observables for ghost fields and antifields ones, see Equation (3.100b). ▽

Remark 3.3.23. This unshifted Poisson structure induces a Poisson structure $\tau : H_\bullet(\mathfrak{Obs}(M)) \otimes H_\bullet(\mathfrak{Obs}(M)) \to \mathbb{R}$ on homology groups. Its component of degree zero $\tau_{00} : H_0(\mathfrak{Obs}(M)) \otimes H_0(\mathfrak{Obs}(M)) \to \mathbb{R}$ coincides with the usual Poisson structure on on-shell gauge-invariant observables which we wrote in Equation (2.53). Remember that the latter coincides with $H_0(\mathfrak{Obs}(M))$. ▽

Remark 3.3.24. By Definition 3.3.20 and Definition 3.3.18 of unshifted Poisson structure and compatible trivializations, respectively, it follows that τ is (graded) antisymmetric with respect to the braiding γ in $\mathsf{Ch}_\mathbb{R}$. It means that it holds $\tau\gamma = -\tau$. This can be proved by relying on the identity $(\mathrm{id} \otimes \Lambda)\gamma = \gamma(\Lambda \otimes \mathrm{id})$:

$$\tau\gamma = 2(-,-)(\mathrm{id} \otimes \Lambda)\gamma = 2(-,-)\gamma(\Lambda \otimes \mathrm{id}) = -\tau\,. \tag{3.101}$$

Thence, the unshifted Poisson structure τ identifies canonically a chain map on the differential graded exterior product $\mathfrak{Obs}(M) \wedge \mathfrak{Obs}(M) := \mathfrak{Obs}(M) \otimes \mathfrak{Obs}(M)/\{v \otimes w = -\gamma(v \otimes w)\}$. With a slight abuse of notation we denote it by the same symbol,

$$\tau : \mathfrak{Obs}(M) \wedge \mathfrak{Obs}(M) \longrightarrow \mathbb{R}\,. \tag{3.102}$$

In other words, we can say that it is a 0-cycle $\tau \in \underline{\mathrm{hom}}\left(\bigwedge^2 \mathfrak{Obs}(M), \mathbb{R}\right)_0$ of the corresponding mapping complex. ▽

There is a last problem we need to consider. The unshifted Poisson structure on $\mathfrak{Obs}(M)$ seems to depend on the choice of the particular pair of compatible trivializations in Definition 3.3.20. Our question is whether different choices of compatible trivializations lead to inequivalent unshifted Poisson structures. The following lemma handle this question.

Lemma 3.3.25. *Suppose $\Lambda^\pm, \widetilde{\Lambda}^\pm$ are two compatible pairs of retarded/advanced trivializations. Denote respectively by $\tau, \widetilde{\tau} \in \underline{\mathrm{hom}}\left(\bigwedge^2 \mathfrak{Obs}(M), \mathbb{R}\right)_0$ the corresponding unshifted Poisson structures. Then there exists a 1-chain $\rho \in \underline{\mathrm{hom}}\left(\bigwedge^2 \mathfrak{Obs}(M), \mathbb{R}\right)_1$ such that $\widetilde{\tau} - \tau = \partial\rho$. This implies that they identify the same homology class: $[\widetilde{\tau}] = [\tau] \in H_0\left(\underline{\mathrm{hom}}\left(\bigwedge^2 \mathfrak{Obs}(M), \mathbb{R}\right)\right)$.*

Proof. Since $\Lambda^\pm, \widetilde{\Lambda}^\pm \in \underline{\mathrm{hom}}\left(\mathfrak{Obs}_{\mathrm{pc/fc}}(M), \mathfrak{Obs}_{\mathrm{pc/fc}}(M)\right)_1$ are retarded/advanced trivializations, from Lemma 3.3.15 we find $\lambda^\pm \in \underline{\mathrm{hom}}\left(\mathfrak{Obs}_{\mathrm{pc/fc}}(M), \mathfrak{Obs}_{\mathrm{pc/fc}}(M)\right)_2$ such that $\widetilde{\Lambda}^\pm - \Lambda^\pm = \partial\lambda^\pm$. Relying on the definition of unshifted Poisson structure, we write:

$$\begin{aligned}
\widetilde{\tau} - \tau &= 2(-,-)\left(\mathrm{id} \otimes \left(j_{\mathrm{pc}}(\widetilde{\Lambda}^+ - \Lambda^+)\iota - j_{\mathrm{fc}}(\widetilde{\Lambda}^- - \Lambda^-)\iota\right)\right) \\
&= 2(-,-)\left(\mathrm{id} \otimes \left(j_{\mathrm{pc}}\partial\lambda^+\iota - j_{\mathrm{fc}}\partial\lambda^-\iota\right)\right) \\
&= \partial\left(2(-,-)\left(\mathrm{id} \otimes \left(j_{\mathrm{pc}}\lambda^+\iota - j_{\mathrm{fc}}\lambda^-\iota\right)\right)\right) = \partial\widetilde{\rho}\,,
\end{aligned} \tag{3.103}$$

where $\widetilde{\rho} \coloneqq 2(-,-)\,(\mathrm{id} \otimes (j_{\mathrm{pc}}\lambda^{+}\iota - j_{\mathrm{fc}}\lambda^{-}\iota)) \in \underline{\mathrm{hom}}\,(\otimes^2\,\mathfrak{Obs}(M), \mathbb{R})_1$. Let us consider the decomposition of $\widetilde{\rho}$ in its (graded) symmetric and antisymmetric parts: $\widetilde{\rho} = \widetilde{\rho}_S + \widetilde{\rho}_A = \frac{1}{2}\,\widetilde{\rho}\,(\mathrm{id} + \gamma) + \frac{1}{2}\,\widetilde{\rho}\,(\mathrm{id} - \gamma)$. Since both $\widetilde{\tau}$ and τ are (graded) antisymmetric, the left-hand side of Equation (3.103) is (graded) antisymmetric and thus its right-hand side must also be such. Therefore, it holds $\widetilde{\tau} - \tau = \partial\widetilde{\rho}_A$ and since $\widetilde{\rho}_A$ is (graded) antysimmetric it defines a 1-chain in $\underline{\mathrm{hom}}\,(\bigwedge^2\,\mathfrak{Obs}(M), \mathbb{R})_1$. □

Therefore, the possible unshifted Poisson structures one can construct all agree up to homotopy. The concrete choice of the advanced/retarded trivializations is not so important in the end. This feature will prove crucial in giving a consistent quantization of our theory.

4 Quantization

The aim of this chapter is to provide a consistent quantization scheme for linearized gravity within the homotopical approach developed so far. We shall begin by constructing the algebra of quantum observables for linearized gravity by imposing the canonical commutation relations (CCR). This construction is proved to be consistent since it preserves quasi-isomorphisms and it sends homotopic Poisson structures introduced in Section 3.3 to weakly equivalent dg-algebras. Then, we shall investigate whether the developed theory is an Algebraic Quantum Field Theory (AQFT). We shall also face the problem of the uniqueness of this quantization prescription, but only a partial result is obtained. Our construction is proved to identify a unique AQFT on each fixed spacetime whilst a non conclusive answer is given for the theory on the entire spacetime category $\mathsf{Loc}_{\mathrm{Ric}}$.

4.1 Quantum observables

Up to now, our study of linearized gravity covered the classical theory with the introduction of the complexes both of the solutions of the equations of motion (Section 3.2) and of the classical observables (Section 3.3). Furthermore, an unshifted Poisson structure, reminiscent of the dynamics, has been introduced. We now want to provide a quantization scheme for this theory.

As an initial step, we want to explain the conceptual framework in which the quantization procedure shall be carried out. It is the so-called Algebraic Quantum Field Theory formalism. To be precise, we shall work with a slight modification of this formalism in order to take into account all the additional information due to our homotopical approach. The algebraic approach to quantum field theory was first suggested by Haag and Kastler in [HK64] and it was then refined by Brunetti, Fredenhagen and Verch in [BFV03]. For the homotopical version of AQFT we refer to [BSW19; BS19a; BBS19].

The algebraic formalism is based on a sharp distinction between the construction of the algebra of quantum observables for a theory and the introduction of a quantum state. Dynamics and causal properties are all encoded in the algebra of observables, whilst all non-local features and correlations are codified in the state.

To fix ideas, let us think about non-relativistic quantum mechanics. The algebraic properties of the observables are encoded in the canonical commutation relation between the position and the momentum operators, q and p respectively, namely $[q, p] = i$.

This commutation relation uniquely identifies the 3 dimensional Heisenberg algebra. Thus, quantum observables are given as soon as their algebraic structure is specified.

Therefore, in order to give a quantization of a (free) field theory, it is first necessary to build a suitable algebra of observables. The kind of algebra one actually needs depends on the context. Having non-relativistic quantum mechanics as a reference, one can realize that an appropriate choice in many situations is to consider a unital algebra $\mathfrak{A}$, over the field of complex numbers $\mathbb{C}$, endowed with a compatible anti-involution $^* : \mathfrak{A} \to \mathfrak{A}$. This is a unital $*$-algebra. There are situations in which it is more convenient to restrict the attention to observables that are bounded (in a suitable sense). In this case, the algebra of observables is often equipped with the structure of a (unital) C^*-algebra. In order to improve readability, we report here the definitions of these structures.

Definition 4.1.1 ($*$-algebra)**.** Let $\mathbb{K}$ be a field equipped with an involution $z \mapsto \bar{z}$. A $*$-*algebra* over $\mathbb{K}$ is a $\mathbb{K}$-vector space $\mathfrak{A}$ equipped with an associative bilinear map $(A, B) \mapsto A \cdot B$ and an anti-linear map $^* : \mathfrak{A} \to \mathfrak{A}$, $A \mapsto A^*$ such that $A^{**} = A$ for all $A \in \mathfrak{A}$ and $(A \cdot B)^* = B^* \cdot A^*$ for all $A, B \in \mathfrak{A}$. The anti-linearity condition means that $(zA + wB)^* = \bar{z}A^* + \bar{w}B^*$ for all $A, B \in \mathfrak{A}$ and $z, w \in \mathbb{K}$. The algebra is *unital* if there exists an element $\mathbb{1} \in \mathfrak{A}$ such that $\mathbb{1} \cdot A = A \cdot \mathbb{1} = A$ for all $A \in \mathfrak{A}$.

Remark 4.1.2. In our case the field $\mathbb{K} = \mathbb{C}$, where the involution is the complex conjugation. $\triangledown$

Definition 4.1.3 (C^*-algebra)**.** A C^*-*algebra* is a $*$-algebra $\mathfrak{A}$ over $\mathbb{C}$ which is also a Banach space whose norm $\|-\| : \mathfrak{A} \to \mathbb{R}$ is compatible with the algebra product and the $*$-involution, *i.e.* it holds $\|A \cdot B\| \leq \|A\|\|B\|$ for all $A, B \in \mathfrak{A}$ and $\|A^* \cdot A\| = \|A\|^2$ for all $A \in \mathfrak{A}$.

The algebra of quantum observables has to capture information about the dynamics of the corresponding physical field and this is achieved by employing the Poisson structure to define the product of the algebra itself. We refer the reader to [BD15] for an analysis in this algebraic formalism of explicit models such as the Klein-Gordon, the Dirac and the Proca fields.

Once the algebra of observables has been defined, a quantum system is identified by its algebraic state.

Definition 4.1.4 (Algebraic state)**.** Let $\mathfrak{A}$ be a unital $(C)^*$-algebra of observables over the field $\mathbb{C}$ of complex numbers. An *algebraic state* on $\mathfrak{A}$ is a linear functional $\rho : \mathfrak{A} \to \mathbb{C}$ such that it is

i. *positive*: $\rho(A^*A)$ is a non-negative real number for all $A \in \mathfrak{A}$;

ii. *normalized*: $\rho(\mathbb{1}) = 1$ for $\mathbb{1} \in \mathfrak{A}$ the unit of the algebra.

Remark 4.1.5. The definition of an algebraic state on the algebra of quantum observables allows us to recover the usual description of Quantum Mechanics. Indeed, the state ρ takes each self-adjoint element of the algebra of observables and it returns its expectation values, measured on that state. In particular, a description in terms of Hilbert spaces and self-adjoint operators on them, where expectation values are computed according to the Born rule, can be recovered via the Gelfand-Naimark-Segal theorem. See [Mor13]. $\triangledown$

In the following we will be mainly concerned with the development of a chain complex analogue of the canonical commutation relations quantization in order to get a suitable notion algebra of observables for linearized gravity. On the contrary, we shall not explore the problem of finding explicit algebraic states for the theory.

Let us start by observing that our homotopical approach in studying the classical theory of linearized gravity leads to a chain complex of linear observables, $\mathfrak{Obs}(M)$, and to an unshifted Poisson structure $\tau : \mathfrak{Obs}(M) \wedge \mathfrak{Obs}(M) \to \mathbb{R}$ on it. For comparison, ordinary field theories lead to vector spaces endowed with Poisson structures as the input for the CCR quantization procedure. Therefore, we need a slight modification of the usual quantization procedure in order to take as an input a pair (V, τ) consisting of a chain complex $V \in \mathsf{Ch}_{\mathbb{R}}$ and of an unshifted Poisson structure $\tau : V \wedge V \to \mathbb{R}$. Nevertheless, it is clear that the output of our construction has to be different. In order to encompass the homotopical machinery introduced while formalizing linearized gravity as a gauge theory, the structure that is more convenient for us is the one of a differential graded unital $*$-algebra over the field $\mathbb{C}$ of complex numbers.

Definition 4.1.6. A *differential graded unital $*$-algebra* $\mathfrak{A}$ over the complex numbers $\mathbb{C}$ is a chain complex equipped with a chain map $\mu : \mathfrak{A} \otimes \mathfrak{A} \to \mathfrak{A}$, a degreewise antilinear involution $^* : \mathfrak{A} \to \mathfrak{A}$ and a unit $\mathbb{1} \in \mathfrak{A}$ such that:

i. μ is associative, *i.e.* $\mu(A \otimes \mu(B \otimes C)) = \mu(\mu(A \otimes B) \otimes C)$ for all $A, B, C \in \mathfrak{A}$;

ii. μ and the $*$-involution are compatible, namely $\mu(A \otimes B)^* = \mu(B^* \otimes A^*)$ for all $A, B \in \mathfrak{A}$;

iii. it holds $\mu(\mathbb{1} \otimes A) = \mu(A \otimes \mathbb{1}) = A$ for each $A \in \mathfrak{A}$.

We are going to explain in some detail how to construct a differential graded unital $*$-algebra $\mathfrak{CCR}(V, \tau)$ that implements the canonical commutation relations generated by the unshifted Poisson structure τ.

First, we build the free differential graded unital $*$-algebra generated by the chain complex $V \in \mathsf{Ch}_{\mathbb{R}}$. This is accomplished by taking the complexification of V, namely $V_{\mathbb{C}} \coloneqq V \otimes \mathbb{C} \in \mathsf{Ch}_{\mathbb{C}}$, and then by defining the complex

$$T^{\otimes}_{\mathbb{C}} V \coloneqq \bigoplus_{n=0}^{\infty} V_{\mathbb{C}}^{\otimes n} \,, \tag{4.1}$$

where we adopt the convention $V_{\mathbb{C}}^{\otimes 0} \coloneqq \mathbb{C}$. The algebra structure is given by juxtaposition

$$\begin{gathered} \mu : T^{\otimes}_{\mathbb{C}} V \otimes T^{\otimes}_{\mathbb{C}} V \longrightarrow T^{\otimes}_{\mathbb{C}} V \,, \\ \mu((v_1 \otimes \cdots \otimes v_n) \otimes (v'_1 \otimes \cdots \otimes v'_m)) \coloneqq v_1 \otimes \cdots \otimes v_n \otimes v'_1 \otimes \cdots \otimes v'_m \,, \end{gathered} \tag{4.2}$$

for all $v_1, \dots, v_n, v'_1, \dots, v'_m \in V$. Observe that μ is a chain map since the tensor product implements the graded Leibniz rule by definition. The unit in $T^{\otimes}_{\mathbb{C}} V$ is given by $\mathbb{1} := 1 \in V^{\otimes 0}_{\mathbb{C}} = \mathbb{C}$. The differential graded unital $*$-algebra structure is completed by the $\mathbb{C}$-antilinear $*$-involution defined by $v^* = v$, for all $v \in V$.

The algebra of observables which implements the canonical commutation relations is constructed by taking the quotient of the free algebra $T^{\otimes}_{\mathbb{C}} V$ by the two-sided differential graded $*$-ideal $\mathcal{I}_{(V,\tau)} \subseteq T^{\otimes}_{\mathbb{C}} V$ generated by the (graded) canonical commutation relations

$$[v_1, v_2] := v_1 \otimes v_2 - (-1)^{|n||m|} v_2 \otimes v_1 = i\tau(v_1, v_2)\mathbb{1}\,, \tag{4.3}$$

for all homogeneous elements $v_1 \in V_n$ and $v_2 \in V_m$. We denote this algebra by

$$\mathfrak{CCR}(V, \tau) := T^{\otimes}_{\mathbb{C}} V / \mathcal{I}_{(V,\tau)}\,. \tag{4.4}$$

Let us make the construction explicit in our case. The CCR algebra of quantum observables for linearized gravity on a physical spacetime M (Definition 2.1.9) is thus given by taking $V = \mathfrak{Obs}(M)$, the linear observable complex from Definition 3.3.1, while τ is an unshifted Poisson structure, as in Definition 3.3.20. We have

$$\mathfrak{CCR}(\mathfrak{Obs}(M), \tau) = T^{\otimes}_{\mathbb{C}} \mathfrak{Obs}(M) / \mathcal{I}_{(\mathfrak{Obs}(M),\tau)}\,. \tag{4.5}$$

Observe that this CCR construction does not identify a unique differential graded unital $*$-algebra of observables for the theory of linearized gravity. Indeed, in principle one gets a different algebra for each possible unshifted Poisson structure on $\mathfrak{Obs}(M)$. We now want to explore whether these different algebras yield in different quantizations of the theory. We shall prove that they are actually all equivalent algebras (in a weak sense that we are going to specify) and in order to do this the result stated in Lemma 3.3.25 will be crucial.

First, let us clarify the categorical setting of our CCR quantization procedure.

Definition 4.1.7. The input and output categories of the CCR quantization procedure depicted above are respectively:

i. the *category of unshifted Poisson complexes* $\mathsf{PoCh}_{\mathbb{R}}$, whose objects are pairs (V, τ) consisting of a chain complex $V \in \mathsf{Ch}_{\mathbb{R}}$ and a chain map $\tau : V \wedge V \to \mathbb{R}$, while morphisms between objects (V, τ) and (V', τ') are chain maps $f : V \to V'$ which preserve the Poisson structure, *i.e.* $\tau'(f \wedge f) = \tau$;

ii. the *category of differential graded unital $*$-algebras* $\mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}$, whose objects are differential graded unital $*$-algebras over the field $\mathbb{C}$ of complex numbers, $(\mathfrak{A}, \mu, ^*)$, while morphisms from $(\mathfrak{A}, \mu, ^*, \mathbb{1})$ to $(\mathfrak{A}', \mu', ^{*'}, \mathbb{1}')$ are chain maps $\kappa : \mathfrak{A} \to \mathfrak{A}'$ which preserve the algebra multiplication, unit and the $*$-involution, *i.e.* $\mu'(\kappa(A) \otimes \kappa(B)) = \kappa(\mu(A \otimes B))$, $\kappa(\mathbb{1}) = \mathbb{1}'$ and $\kappa(A)^{*'} = \kappa(A^*)$ for all $A, B \in \mathfrak{A}$.

Lemma 4.1.8. *The CCR quantization procedure identifies a functor*

$$\mathfrak{CCR} : \mathsf{PoCh}_{\mathbb{R}} \longrightarrow \mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}\,. \tag{4.6}$$

Proof. The CCR quantization associates an object $\mathfrak{CCR}(V,\tau) \in \mathsf{dg^*Alg}_{\mathbb{C}}$ to each unshifted Poisson complex $(V,\tau) \in \mathsf{PoCh}_{\mathbb{R}}$. Therefore, the action of $\mathfrak{CCR}$ on the objects of the category is well-posed. As far as the morphisms are concerned, we have to make clear how $\mathfrak{CCR}$ acts on them. Let $f \in \mathsf{PoCh}_{\mathbb{R}}((V,\tau),(V',\tau'))$. Then, $\mathfrak{CCR}(f)$ has to be a $\mathsf{dg^*Alg}_{\mathbb{C}}$-morphism between $\mathfrak{CCR}(V,\tau)$ and $\mathfrak{CCR}(V',\tau')$. We set $\mathfrak{CCR}(f) \coloneqq \bigoplus_{n=0}^{\infty} f_{\mathbb{C}}^{\otimes n}$, where with $f_{\mathbb{C}}$ we mean the natural extension of f to a $\mathbb{C}$-linear map. It remains to be checked that it defines a morphism and that this association is functorial. The map $\mathfrak{CCR}(f)$ is well-defined since it preserves the canonical commutation relations (4.3):

$$\begin{aligned}\mathfrak{CCR}(f)\,[v_1,v_2] &= f(v_1)\otimes f(v_2) - (-1)^{|n||m|} f(v_2)\otimes f(v_1) = [f(v_1),f(v_2)]' \\ &= i\tau'(f(v_1)\otimes f(v_2))\mathbb{1} = i\tau(v_1,v_2)\mathbb{1} = [v_1,v_2]\ , \end{aligned} \tag{4.7}$$

for each $v_1 \in V_n$ and $v_2 \in V_m$, where the second last step follows because f is a $\mathsf{PoCh}_{\mathbb{R}}$-morphism. Furthermore, it is clearly a chain map and it is compatible with the $*$-involution since, by definition, the latter coincides with the identity on elements of both V and V'. Finally, $\mathfrak{CCR}(f)$ is by construction compatible with multiplication. To see functoriality of $\mathfrak{CCR}$ we note that $\mathfrak{CCR}(\mathrm{id}) = \mathrm{id}$ and $\mathfrak{CCR}(f\circ g) = \mathfrak{CCR}(f)\circ\mathfrak{CCR}(g)$ for any f,g composable $\mathsf{PoCh}_{\mathbb{R}}$-morphisms. Both of them are direct consequences of the definition of $\mathfrak{CCR}$ and of tensor product properties. □

In order to study the homotopical properties of our CCR quantization we need to endow both $\mathsf{PoCh}_{\mathbb{R}}$ and $\mathsf{dg^*Alg}_{\mathbb{C}}$ with the structure of a *homotopical category*.

Definition 4.1.9. A *homotopical category* is the datum of a category C and a class of morphisms $\mathcal{W}$ such that

i. every identity morphism id_C, for $C \in \mathsf{C}$, is in $\mathcal{W}$;

ii. the 2-*out-of-*6 *property* is satisfied, *i.e.* if $h\circ g, g\circ f \in \mathcal{W}$ then also $f,g,h,h\circ g\circ f \in \mathcal{W}$.

Morphisms in $\mathcal{W}$ are called *weak equivalences*.

Remark 4.1.10. The definition of weak equivalences of a homotopical category is such that all isomorphisms are weak equivalences. Indeed, if $f \in \mathsf{C}(A,B)$ is an isomorphism, there exists $f^{-1} \in \mathsf{C}(B,A)$ such that $f\circ f^{-1} = \mathrm{id}_B$ and $f^{-1}\circ f = \mathrm{id}_A$. By definition, both are weak equivalences and 2-out-of-6 property implies, in particular, $f \in \mathcal{W}$. ▽

Observe that this homotopical structure is more flexible than that of the model category. It requires only to declare a class of weak equivalences without bothering to introduce compatible classes of fibrations and cofibrations or requiring the category to be bicomplete.

Definition 4.1.11. The categories $\mathsf{PoCh}_{\mathbb{R}}$ and $\mathsf{dg^*Alg}_{\mathbb{C}}$ provide homotopical categories with the following choices of weak equivalences:

i. A morphism $f : (V,\tau) \to (V',\tau')$ in $\mathsf{PoCh}_{\mathbb{R}}$ is a *weak equivalence* if the corresponding chain map $f : V \to V'$ is a quasi-isomorphism;

ii. A morphism $\kappa : (\mathfrak{A}, \mu, ^*) \to (\mathfrak{A}', \mu', ^{*'})$ in $\mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}$ is a *weak equivalence* if its underlying chain map $\kappa : \mathfrak{A} \to \mathfrak{A}'$ is a quasi-isomorphism.

Remark 4.1.12. It may not be immediate to see that these definitions actually provide classes of weak equivalences in the sense of Definition 4.1.9. In particular the 2-out-of-6 property should be checked. However, it can be shown by relying on the model structure of the category $\mathsf{Ch}_{\mathbb{K}}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, depending on the case. In fact, each morphism which has been declared to be a weak equivalence in Definition 4.1.11 is a weak equivalence also when it is seen as a morphism in the model category $\mathsf{Ch}_{\mathbb{K}}$. Furthermore, the 2-out-of-6 property is satisfied by all the weak equivalences of a model category. $\triangledown$

Thanks to the homotopical category machinery, we are able to prove that the CCR quantization functor of Equation (4.6) has such homotopical properties that allow us to give a consistent quantization of linearized gravity on a spacetime M. The main result we are going to use is the following proposition proved in [BBS19]. We report here its proof for the sake of completeness.

Proposition 4.1.13. *Let* $\mathfrak{CCR} : \mathsf{PoCh}_{\mathbb{R}} \to \mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}$ *be the CCR quantization functor of Lemma 4.1.8. It is a homotopical functor, when the domain and the target categories are endowed with the homotopical structures from Definition 4.1.11,* i.e. *it sends weak equivalences of* $\mathsf{PoCh}_{\mathbb{R}}$ *to weak equivalences of* $\mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}$*. Moreover, if* $(V, \tau) \in \mathsf{PoCh}_{\mathbb{R}}$ *is an unshifted Poisson complex and* $\rho \in \underline{\mathrm{hom}}\,(\bigwedge^2 V, \mathbb{R})_1$ *is a 1-chain, then there exists a zigzag*

$$\mathfrak{CCR}(V,\tau) \xleftarrow{\;\sim\;} A_{(V,\tau,\rho)} \xrightarrow{\;\sim\;} \mathfrak{CCR}(V, \tau + \partial\rho) \tag{4.8}$$

of weak equivalences in $\mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}$.

Proof. Consider the homotopical category of *differential graded unital Lie $*$-algebras* $\mathsf{dg}^*\mathsf{uLie}_{\mathbb{C}}$, where weak equivalences are morphisms whose underlying chain map is a quasi-isomorphism. It is proved in [BS19b] that the $\mathfrak{CCR}$ functor (4.6) admits a factorization through $\mathsf{dg}^*\mathsf{uLie}_{\mathbb{C}}$:

$$\begin{array}{ccc} \mathsf{PoCh}_{\mathbb{R}} & \xrightarrow{\quad\mathfrak{CCR}\quad} & \mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}} \\ & {\scriptstyle\mathfrak{Heis}}\searrow \quad \nearrow{\scriptstyle\mathfrak{U}_{\mathrm{lin}}} & \\ & \mathsf{dg}^*\mathsf{uLie}_{\mathbb{C}} & \end{array} \tag{4.9}$$

The functor $\mathfrak{U}_{\mathrm{lin}} : \mathsf{dg}^*\mathsf{uLie}_{\mathbb{C}} \to \mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}$ is the unital universal enveloping algebra functor, which is proved to be homotopical in [BS19b]. The Heisenberg Lie algebra functor $\mathfrak{Heis} : \mathsf{PoCh}_{\mathbb{R}} \to \mathsf{dg}^*\mathsf{uLie}_{\mathbb{C}}$ is then explicitly given, for $(V, \tau) \in \mathsf{PoCh}_{\mathbb{R}}$, by

$$\mathfrak{Heis}(V,\tau) := V_{\mathbb{C}} \oplus \mathbb{C}\,, \tag{4.10}$$

with Lie bracket $[-,-] : (V_{\mathbb{C}} \oplus \mathbb{C}) \otimes (V_{\mathbb{C}} \oplus \mathbb{C}) \to V_{\mathbb{C}} \oplus \mathbb{C}$ defined by

$$[v_1 \oplus z_1, v_2 \oplus z_2] := 0 \oplus i\tau(v_1, v_2)\,, \tag{4.11}$$

for $v_1, v_2 \in V$ and $z_1, z_2 \in \mathbb{C}$. Finally, the unit is $\mathbb{1} := 0 \oplus 1$ and the $*$-involution is determined by $v^* = v$ for all $v \in V$ and by complex conjugation on the $\mathbb{C}$ component. To a $\mathsf{PoCh}_{\mathbb{R}}$-morphism $f : (V, \tau) \to (V', \tau')$ it is associated the morphism $\mathfrak{Heis}(f) : \mathfrak{Heis}(V, \tau) \to \mathfrak{Heis}(V', \tau')$ in $\mathsf{dg}^*\mathsf{uLie}_{\mathbb{C}}$ determined by $f_{\mathbb{C}} \oplus \mathrm{id} : V_{\mathbb{C}} \oplus \mathbb{C} \to V'_{\mathbb{C}} \oplus \mathbb{C}$, where we recall that $f_{\mathbb{C}}$ is the complexification of the chain map f. Also the Heisenberg Lie algebra functor is homotopical too. It is proved in [BBS19] by observing that both the operations $(-) \otimes \mathbb{C}$ and $(-) \oplus \mathbb{C}$ preserve quasi-isomorphisms and this implies immediately that $f_{\mathbb{C}} \oplus \mathrm{id}$ is a quasi-isomorphism once f is a quasi-isomorphism. Since $\mathfrak{CCR}$ results to be a composition of functors that preserve weak equivalences, it is a homotopical functor and the first part of the proposition is proved.

Regarding the second part of the statement, it can be proved by showing that a very analogous result holds true with respect to the Heisenberg Lie algebra functor $\mathfrak{Heis}$. The sought result follows then due to the homotopical properties of $\mathfrak{Q}_{\mathrm{lin}}$ and Equation (4.9). Consequentely, we need to find a zigzag of weak equivalences in $\mathsf{dg}^*\mathsf{uLie}_{\mathbb{C}}$,

$$\mathfrak{Heis}(V, \tau) \xleftarrow{\sim} H_{(V,\tau,\rho)} \xrightarrow{\sim} \mathfrak{Heis}(V, \tau + \partial\rho)\,, \tag{4.12}$$

for all $(V, \tau) \in \mathsf{PoCh}_{\mathbb{R}}$ and $\rho \in \underline{\mathrm{hom}}\left(\bigwedge^2 V, \mathbb{R}\right)_1$. The object $H_{(V,\tau,\rho)} \in \mathsf{dg}^*\mathsf{uLie}_{\mathbb{C}}$ reads explicitly

$$H_{(V,\tau,\rho)} := V_{\mathbb{C}} \oplus D \oplus \mathbb{C}\,, \tag{4.13}$$

where $D := \left(\overset{(-1)}{\mathbb{C}} \xleftarrow{\mathrm{id}} \overset{(0)}{\mathbb{C}} \right)$ is the "disk" chain complex. The unital Lie structure is completed by the unit $\mathbb{1} := 0 \oplus 0 \oplus 1$ and by the Lie bracket

$$[v_1 \oplus \alpha_1 \oplus z_1, v_2 \oplus \alpha_2 \oplus z_2] := 0 \oplus (i\partial\rho(v_1, v_2)x + i\rho(v_1, v_2)y) \oplus i\tau(v_1, v_2)\,, \tag{4.14}$$

where $x := 1 \in D_0$ and $y := 1 \in D_{-1}$. Let $s \in \mathbb{R}$ and denote with $\mathcal{I}_s \subseteq H_{(V,\tau,\rho)}$ the differential graded unital Lie $*$-algebra ideal generated by the relations

$$0 \oplus x \oplus 0 = 0 \oplus 0 \oplus s\,, \quad 0 \oplus y \oplus 0 = 0\,. \tag{4.15}$$

Observe that the quotient algebra $H_{(V,\tau,\rho)}/\mathcal{I}_s$ is isomorphic to $\mathfrak{Heis}(V, \tau + s\partial\rho)$. Finally, consider the quotient map

$$\pi_s : H_{(V,\tau,\rho)} \longrightarrow H_{(V,\tau,\rho)}/\mathcal{I}_s \cong \mathfrak{Heis}(V, \tau + s\partial\rho)\,. \tag{4.16}$$

By observing the relations (4.15), one realizes that $\pi_s = \mathrm{id}_V \oplus q_s$, where $q_s : D \oplus \mathbb{C} \to \mathbb{C}$ is the map given by $q_s\left((z_1 x + z_2 y) \oplus z_3\right) := s z_1 + z_3$. Therefore, the quotient map π_s is a quasi-isomorphism for any $s \in \mathbb{R}$ since the induced map on the homologies is nothing more than identity. The zigzag in Equation (4.12) is finally obtained by taking $s = 0$ and $s = 1$. □

Corollary 4.1.14. *Suppose* $\Lambda^{\pm}, \widetilde{\Lambda}^{\pm} \in \underline{\mathrm{hom}}\left(\mathfrak{Obs}_{\mathrm{pc/fc}}(M), \mathfrak{Obs}_{\mathrm{pc/fc}}(M)\right)_1$ *are two compatible pairs of retarded/advanced trivializations and denote the corresponding unshifted Poisson structures by* $\tau, \widetilde{\tau} : \mathfrak{Obs}(M) \wedge \mathfrak{Obs}(M) \to \mathbb{R}$. *Then their two CCR quantizations* $\mathfrak{CCR}(\mathfrak{Obs}(M), \tau) \simeq \mathfrak{CCR}(\mathfrak{Obs}(M), \widetilde{\tau})$ *are weakly equivalent via a zigzag of weak equivalences in* $\mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}$.

Proof. This is a direct consequence of Proposition 4.1.13 and Lemma 3.3.25. Let $\Lambda^{\pm}, \widetilde{\Lambda}^{\pm}$ be two pairs of compatible trivializations, then from Lemma 3.3.25 it follows that the corresponding unshifted Poisson structures belong to the same homology class. Hence, there exists a 1-chain $\rho \in \underline{\mathrm{hom}}\left(\bigwedge^2 \mathfrak{Obs}(M), \mathbb{R}\right)_1$ such that $\widetilde{\tau} = \tau + \partial\rho$. The corresponding CCR quantizations are $\mathfrak{CCR}(\mathfrak{Obs}(M), \tau)$ and $\mathfrak{CCR}(\mathfrak{Obs}(M), \tau + \partial\rho)$. Finally, Proposition 4.1.13 guarantees the existence of a zigzag

$$\mathfrak{CCR}(\mathfrak{Obs}(M), \tau) \xleftarrow{\sim} A_{(\mathfrak{Obs}(M),\tau,\rho)} \xrightarrow{\sim} \mathfrak{CCR}(\mathfrak{Obs}(M), \tau + \partial\rho) \tag{4.17}$$

of weak equivalences in $\mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}$. □

Remark 4.1.15. Corollary 4.1.14 guarantees that our quantization prescription for linearized gravity is consistent. Indeed, the CCR quantization which corresponds to our choice of retarded/advanced trivializations from Proposition 3.3.14 is equivalent via a zigzag of weak equivalences in $\mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}$ to the quantization that one gets by selecting any other pair of compatible trivializations, see also Corollary 3.3.16. ▽

We conclude this section by giving an explicit characterization of the quantum observable algebra corresponding to the unshifted Poisson structure τ of Corollary 3.3.21. Observe that elements in the differential graded unital $*$-algebra $\mathfrak{CCR}(\mathfrak{Obs}(M), \tau)$ from Equation (4.5) are generated by elements in the first component of the direct sum (4.1), *i.e.* $\mathfrak{Obs}(M)_{\mathbb{C}}$. Hence, we can introduce the following evocative notation for the generators of the CCR algebra: We denote the smeared linear quantum observables for gauge fields by $\widehat{h}(\varepsilon)$ for $\varepsilon \in \mathfrak{Obs}(M)_0 = \Gamma_c(\otimes_S^2 T^*M)$, the ones for ghost fields by $\widehat{\chi}(\eta)$ for $\eta \in \mathfrak{Obs}(M)_{-1} = \Gamma_c(T^*M)$, and the ones for antifields by $\widehat{h}^{\ddagger}(\alpha)$ and $\widehat{\chi}^{\ddagger}(\beta)$ for $\alpha \in \mathfrak{Obs}(M)_1 = \Gamma_c(\otimes_S^2 T^*M)$ and $\beta \in \mathfrak{Obs}(M)_2 = \Gamma_c(T^*M)$. By making explicit use of the unshifted Poisson structure τ of Equations (3.100) in the (graded) commutator of Equation (4.3), we find the following non-vanishing (graded) commutation relations between algebra generators:

$$\left[\widehat{h}(\varepsilon_1), \widehat{h}(\varepsilon_2)\right] = -2i \int_M (\varepsilon_1)_{ab} (G\varepsilon_2)_{cd} g^{ac} g^{bd} \mu_g \, \mathbb{1} \,, \tag{4.18a}$$

$$\left[\widehat{h}^{\ddagger}(\alpha), \widehat{\chi}(\eta)\right] = -4i \int_M (\alpha)_{ab} (G\nabla_S \eta)_{cd} g^{ac} g^{bd} \mu_g \, \mathbb{1} = \left[\widehat{\chi}(\eta), \widehat{h}^{\ddagger}(\alpha)\right] \,, \tag{4.18b}$$

for all $\varepsilon_1, \varepsilon_2 \in \mathfrak{Obs}(M)_0 = \Gamma_c(\otimes_S^2 T^*M)$, $\alpha \in \mathfrak{Obs}(M)_1 = \Gamma_c(\otimes_S^2 T^*M)$ and $\eta \in \mathfrak{Obs}(M)_{-1} = \Gamma_c(T^*M)$.

4.2 Homotopy AQFT axioms and linearized gravity

So far we have developed a quantization prescription for linearized gravity on a fixed globally hyperbolic Ricci-flat Lorentzian manifold M. In the context of the algebraic formalism, both in the sense of Haag and Kastler [HK64] and of Brunetti, Fredenhagen and Verch [BFV03], the notion of a quantum field theory is related to a functorial assignment with respect to a suitable class of spacetime embeddings. To be more explicit, an Algebraic Quantum Field Theory is a functor that assigns the algebra of quantum observables, belonging to a suitable category Alg, to each spacetime, object of a suitable category C,

$$\mathfrak{A} : \mathsf{C} \longrightarrow \mathsf{Alg}\,. \tag{4.19}$$

The functor $\mathfrak{A}$ satisfies certain axioms imposed due to physical reasons:

- The spacetime category C is endowed with a consistent notion of "causally disjoint" morphisms. This is correctly implemented by *orthogonal categories* as introduced in [BSW20];

- A typical condition on the AQFT functor $\mathfrak{A}$ is that it maps causally disjoint morphisms into commuting ones. This is the so-called *Einstein causality axiom* and it captures the idea that causally disjoint observables have to commute with each other;

- Another common axiom is the *time-slice axiom* which encodes a concept of time evolution by demanding an equivalence between the algebra corresponding to the full spacetime and those of suitable subspaces of it.

We will come back to this topic later in this section and we will state these axioms more accurately, adapting them to the homotopical framework we are working with.

We start by introducing the relevant categories which will appear in our construction.

Definition 4.2.1. The category Alg of the algebras of observables in Equation (4.19) coincides with that of differential graded unital $*$-algebras $\mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}$ of Definition 4.1.7. The background spacetime category C may refer to different categories. Depending on the context, we have $\mathsf{C} = \mathsf{Loc}_{\mathrm{Ric}}$ or $\mathsf{C} = \mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$, for $\overline{M} \in \mathsf{Loc}_{\mathrm{Ric}}$, where

- $\mathsf{Loc}_{\mathrm{Ric}}$ denotes the category of oriented and time-oriented globally hyperbolic Ricci-flat Lorentzian manifolds of dimension $n = 4$ with morphisms $f : M \to N$ given by all orientation and time-orientation preserving isometric embeddings whose image $f(M) \subseteq N$ is open and causally convex.

- For any fixed $\overline{M} \in \mathsf{Loc}_{\mathrm{Ric}}$ we denote by $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$ the corresponding slice category. Its objects are all morphisms $m : M \to \overline{M}$ with target $\overline{M}$ in $\mathsf{Loc}_{\mathrm{Ric}}$ and its morphisms $f : (m : M \to \overline{M}) \to (n : N \to \overline{M})$ are all commutative triangles

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & N \\ & {\scriptstyle m}\searrow \quad \swarrow{\scriptstyle n} & \\ & \overline{M} & \end{array} \tag{4.20}$$

in $\mathsf{Loc}_{\mathrm{Ric}}$. The slice category $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$ for $\overline{M} \in \mathsf{Loc}_{\mathrm{Ric}}$ is equivalent to the category of all causally convex open subsets $U \subseteq \overline{M}$ with morphisms given by subset inclusion.

Since all the relevant categories have been introduced we shall now give a precise definition of a homotopy AQFT by making explicit the homotopy version of the AQFT axioms we need. Here we follow the formulation of homotopy AQFTs presented in [BBS19] for a setting very akin to ours. This is a particular case of a more general operadic approach to AQFTs developed in [BSW19; BSW20].

Definition 4.2.2. A *homotopy AQFT* on $\mathsf{Loc}_{\mathrm{Ric}}$ is a functor $\mathfrak{A} : \mathsf{Loc}_{\mathrm{Ric}} \to \mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}$ such that the following axioms hold true:

i. *Einstein causality:* For every pair $(f_1 : M_1 \to N, f_2 : M_2 \to N)$ of $\mathsf{Loc}_{\mathrm{Ric}}$-morphisms with causally disjoint images in N, the chain map

$$[\mathfrak{A}(f_1)(-), \mathfrak{A}(f_2)(-)] : \mathfrak{A}(M_1) \otimes \mathfrak{A}(M_2) \longrightarrow \mathfrak{A}(N) \tag{4.21}$$

vanishes, where $[-,-] \coloneqq \mu - \mu\gamma : \mathfrak{A}(N) \otimes \mathfrak{A}(N) \to \mathfrak{A}(N)$ is the (graded) commutator in $\mathfrak{A}(N)$.

ii. *Time-slice:* For every Cauchy morphism, *i.e.* a $\mathsf{Loc}_{\mathrm{Ric}}$-morphism $f : M \to N$ whose image $f(M) \subseteq N$ contains a Cauchy surface of N, the map $\mathfrak{A}(f) : \mathfrak{A}(M) \to \mathfrak{A}(N)$ is a weak equivalence in the homotopical category $\mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}$.

Remark 4.2.3. A *homotopy AQFT on* $\overline{M} \in \mathsf{Loc}_{\mathrm{Ric}}$ is given by replacing all the $\mathsf{Loc}_{\mathrm{Ric}}$ occurrences in Definition 4.2.2 with the slice category $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$. ▽

Remark 4.2.4. Observe that all homotopy AQFTs on $\mathsf{Loc}_{\mathrm{Ric}}$ are chain complex analogues of theories in the sense of Brunetti, Fredenhagen and Verch [BFV03], while homotopy AQFTs in the sense of Remark 4.2.3 are theories on a fixed physical spacetime $\overline{M} \in \mathsf{Loc}_{\mathrm{Ric}}$ and are chain complex analogues of theories in the sense of Haag and Kastler [HK64]. Note that every homotopy AQFT $\mathfrak{A} : \mathsf{Loc}_{\mathrm{Ric}} \to \mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}$ on $\mathsf{Loc}_{\mathrm{Ric}}$ also identifies a theory on a fixed $\overline{M} \in \mathsf{Loc}_{\mathrm{Ric}}$. It is given by the functor $\mathfrak{A}_{\overline{M}} \coloneqq \mathfrak{A}\,\mathfrak{U}_{\overline{M}} : \mathsf{Loc}_{\mathrm{Ric}}/\overline{M} \to \mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}$ obtained via precomposition with the forgetful functor $\mathfrak{U}_{\overline{M}} : \mathsf{Loc}_{\mathrm{Ric}}/\overline{M} \to \mathsf{Loc}_{\mathrm{Ric}}$. Explicitly, the latter is given on objects by $(m : M \to \overline{M}) \mapsto M$ and on morphisms by $(f : (m : M \to \overline{M}) \to (n : N \to \overline{M})) \mapsto (f : M \to N)$. ▽

We consider now all constructions that we encountered in our reasonings and analyze their functoriality on $\mathsf{Loc}_{\mathrm{Ric}}$. Let us start from the assignment of vector bundles to a background manifold M. In Equation (3.14) the only vector bundles that appear are the cotangent bundle T^*M and the bundle of completely symmetric covariant 2-tensors $\otimes^2_S T^*M$, together with integral pairings as per Equation (2.28). For notational simplicity we will use $F(M)$ to denote both these bundles on a manifold M. Therefore, $\Gamma F(M)$ will be the space of smooth sections of the bundle $F(M)$. In order to see the (contravariant) functoriality of the bundle F one has to check that the assignment $\Gamma F : {\mathsf{Loc}_{\mathrm{Ric}}}^{\mathrm{op}} \to \mathsf{Vec}$, $M \mapsto \Gamma F(M)$ is functorial. Hence, we need to prove that each $\mathsf{Loc}_{\mathrm{Ric}}$-morphism $f : M \to N$ lifts to a linear morphism $f^* : \Gamma F(N) \to \Gamma F(M)$. This holds true for f^*, the pullback between covariant tensor fields,

$$(f^*t)_P(v_1, \ldots, v_k) \coloneqq t_{f(P)}(f_*v_1, \ldots, f_*v_k)\,, \tag{4.22}$$

for all $t \in \Gamma(\otimes^k T^*N)$, $P \in M$ and $v_i \in T_P M$, for $i = 1, \dots, k$. Here $f_* : T_P M \to T_{f(P)}N$ denotes the pushforward of tangent vectors along the morphism f, namely, for $v \in T_P M$, $f_* v \in T_{f(P)}N$ is

$$(f_* v)(\phi) := v(\phi \circ f)\,, \tag{4.23}$$

for all smooth functions $\phi : N \to \mathbb{R}$. Note that in Equation (4.22) the covariant k-tensor t is seen as a multilinear map on k copies of the tangent space TN. Furthermore, the integral pairings are preserved under pullbacks since f is an isometry per hypothesis.

Definition 4.2.5. Let $\Gamma F, \Gamma G : \mathsf{Loc}_{\mathrm{Ric}}{}^{\mathrm{op}} \to \mathsf{Vec}$ be two natural spaces of sections of vector bundles on $\mathsf{Loc}_{\mathrm{Ric}}$. Then a *natural operator* between them is a natural transformation $K : \Gamma F \Rightarrow \Gamma G$.

The following lemma turns out to be a crucial tool in proving the naturality of the operators which appear in the chain complexes (3.14), (3.39) and (3.65).

Lemma 4.2.6. *Suppose (M, g_M) and (N, g_N) are Lorentzian manifolds. Denote by ∇^M and ∇^N the corresponding Levi-Civita connections. Then, for any isometric embedding $f : M \to N$ the identity*

$$\nabla^M (f^* r) = f^*(\nabla^N r) \tag{4.24}$$

holds true for all $r \in \Gamma(\otimes^k T^*N)$.

Proof. Since f is an embedding the image $f(M) \subseteq N$ is a submanifold of N, diffeomorphic to M. Moreover, if $\widetilde{g} := g_N\big|_{f(M)}$ is the restriction on submanifold $f(M)$ of the metric on N, the corestriction of f on its image, $\widetilde{f} := f\big|^{f(M)}$, is an isometric diffeomorphism between the Lorentzian manifolds (M, g_M) and $(f(M), \widetilde{g})$. Denote by $\widetilde{\nabla}$ the Levi-Civita connection on $f(M)$ with respect to the metric $\widetilde{g}$. Since $\widetilde{f}$ is an isometric diffeomorphism and relying on the uniqueness of Levi-Civita connection it is possible to show that the identity $\widetilde{f}_* \nabla^M \widetilde{f}^* = \widetilde{\nabla}$ holds true. Finally, let us consider the set inclusion $i : f(M) \to N$. Clearly it holds $f = i \circ \widetilde{f}$ and hence $f^* = \widetilde{f}^* \circ i^*$. Moreover, the fact that i is a set inclusion entails that $i^* \nabla^N = \widetilde{\nabla} i^*$ on $\Gamma(\otimes^k T^*N)$. Then, the chain of identities,

$$\nabla^M f^* = \nabla^M \widetilde{f}^* i^* = \widetilde{f}^* \widetilde{\nabla} i^* = \widetilde{f}^* i^* \nabla^N = f^* \nabla^N\,, \tag{4.25}$$

concludes the proof. $\square$

Proposition 4.2.7. *The operators $(\nabla_S^M : \Gamma(T^*M) \to \Gamma(\otimes_S^2 T^*M))_{M \in \mathsf{Loc}_{\mathrm{Ric}}}$ and $(P^M : \Gamma(\otimes_S^2 T^*M) \to \Gamma(\otimes_S^2 T^*M))_{M \in \mathsf{Loc}_{\mathrm{Ric}}}$ are natural.*

Proof. We need to prove that ∇_S and P induce natural transformations between the suitable spaces of sections. Let $f : M \to N$ be a $\mathsf{Loc}_{\mathrm{Ric}}$-morphism. The naturality condition for ∇_S translates to the commutativity of the square below.

$$\begin{array}{ccc} \Gamma(T^*M) & \xleftarrow{\;f^*\;} & \Gamma(T^*N) \\ {\scriptstyle \nabla_S^M}\big\downarrow & & \big\downarrow{\scriptstyle \nabla_S^N} \\ \Gamma(\otimes_S^2 T^*M) & \xleftarrow[\;f^*\;]{} & \Gamma(\otimes_S^2 T^*N) \end{array} \tag{4.26}$$

This is a direct consequence of Lemma 4.2.6 since the Killing operator ∇_S is defined as the normalized symmetrization of the covariant derivative ∇. Concerning the dynamical operator P, we recall that it reads $P = (-\operatorname{div}\nabla + 2\operatorname{Riem} + 2I\nabla_S \operatorname{div})\, I$. Since $f : M \to N$ is an isometric embedding, taking the trace with respect to the metric is a natural operation and, as a consequence, the trace reversal $I^M : \Gamma(\otimes^2_S T^*M) \to \Gamma(\otimes^2_S T^*M)$ also identifies a natural operator. This, together with the naturality of the Levi-Civita connection and of the Riemann operator, $\operatorname{Riem}(f^*g_N) = f^*\operatorname{Riem}(g_N)$, implies that the identity $f^*P = Pf^*$ holds true and thus naturality of the dynamical operator P is proved. □

Naturality of the cotangent bundle T^*M and of the symmetric 2-tensor bundle $\otimes^2_S T^*M$, together with that of the differential operator ∇_S which encodes the action of ghost fields on gauge fields, entails that the assignment $M \mapsto \mathfrak{C}(M)$ of linearized gravity field complexes as per Equation (3.14) is contravariantly functorial, *i.e.*

$$\mathfrak{C} : \mathsf{Loc_{Ric}}^{\mathrm{op}} \longrightarrow \mathsf{Ch}_{\mathbb{R}} \,. \tag{4.27}$$

Observe that the commutativity of the square (4.26) is equivalent to saying that the pullback f^* of any $\mathsf{Loc_{Ric}}$-morphism $f : M \to N$ extends to a chain map $f^* : \mathfrak{C}(N) \to \mathfrak{C}(M)$.

Since the dynamical operator P is natural, as stated in Proposition 4.2.7, also the assignment $M \mapsto \mathfrak{Sol}(M)$ of solution complexes as per Equation (3.39) is contravariantly functorial, *i.e.*

$$\mathfrak{Sol} : \mathsf{Loc_{Ric}}^{\mathrm{op}} \longrightarrow \mathsf{Ch}_{\mathbb{R}} \,. \tag{4.28}$$

In order to deal with the functoriality of the assignment of linear observable complexes we need to consider also the pushforwards $f_* : \Gamma_c(FM) \to \Gamma_c(FN)$, with either $F(-) = T^*(-)$ or $F(-) = \otimes^2_S T^*(-)$, of compactly supported sections along $\mathsf{Loc_{Ric}}$-morphisms $f : M \to N$. By exploiting them, one realizes that the assignment $M \mapsto \mathfrak{Obs}(M)$ is functorial, *i.e.*

$$\mathfrak{Obs} : \mathsf{Loc_{Ric}} \longrightarrow \mathsf{Ch}_{\mathbb{R}} \,, \tag{4.29}$$

where the action of the functor $\mathfrak{Obs}$ on $\mathsf{Loc_{Ric}}$-morphisms $f : M \to N$ is given explicitly by pushforwards on each degree, $\mathfrak{Obs}(f) = f_* : \mathfrak{Obs}(M) \to \mathfrak{Obs}(N)$.

Finally, let us consider naturality of the integral pairings, as per Equation (3.66), and of the shifted Poisson structure from Definition 3.3.8. The pairings are natural in the sense that the square

$$\begin{array}{ccc} \mathfrak{Obs}(M) \otimes \mathfrak{Sol}(N) & \xrightarrow{f_*\otimes \mathrm{id}} & \mathfrak{Obs}(N) \otimes \mathfrak{Sol}(N) \\ {\scriptstyle \mathrm{id}\otimes f^*}\big\downarrow & & \big\downarrow{\scriptstyle (-,-)^N} \\ \mathfrak{Obs}(M) \otimes \mathfrak{Sol}(M) & \xrightarrow[(-,-)^M]{} & \mathbb{R} \end{array} \tag{4.30}$$

commutes for all $f : M \to N$ morphisms in $\mathsf{Loc_{Ric}}$. Furthermore, the chain map j of

Equation (3.80) gives rise to the following commutative diagram

$$\begin{array}{ccc} \mathfrak{Obs}(M) & \xrightarrow{f_*} & \mathfrak{Obs}(N) \\ {\scriptstyle j^M}\downarrow & & \downarrow{\scriptstyle j^N} \\ \mathfrak{Sol}(M)[1] & \xleftarrow[f^*]{} & \mathfrak{Sol}(N)[1] \end{array} \tag{4.31}$$

for all $\mathsf{Loc}_{\mathrm{Ric}}$-morphisms $f : M \to N$. By combining squares (4.30) and (4.31) and by making use of Definition 3.3.8, one observes that the shifted Poisson structure Υ is natural in the sense that the square

$$\begin{array}{ccc} \mathfrak{Obs}(M) \otimes \mathfrak{Obs}(M) & \xrightarrow{\Upsilon^M} & \mathbb{R}[1] \\ {\scriptstyle f_* \otimes f_*}\downarrow & & \Vert \\ \mathfrak{Obs}(N) \otimes \mathfrak{Obs}(N) & \xrightarrow[\Upsilon^N]{} & \mathbb{R}[1] \end{array} \tag{4.32}$$

commutes for all $f : M \to N$ $\mathsf{Loc}_{\mathrm{Ric}}$-morphisms.

Remark 4.2.8. All functors and natural transformations on $\mathsf{Loc}_{\mathrm{Ric}}$ that we have introduced so far admit restrictions to the slice category $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$ for each $\overline{M} \in \mathsf{Loc}_{\mathrm{Ric}}$. They are obtained via precomposition with the forgetful functor $\mathfrak{U}_{\overline{M}} : \mathsf{Loc}_{\mathrm{Ric}}/\overline{M} \to \mathsf{Loc}_{\mathrm{Ric}}$ introduced in Remark 4.2.4. These restrictions are sufficient if one is interested in the construction of a homotopy AQFT on a fixed spacetime $\overline{M} \in \mathsf{Loc}_{\mathrm{Ric}}$, as opposed to a theory on the entire category $\mathsf{Loc}_{\mathrm{Ric}}$. ▽

Our construction of a quantum observable algebra for any fixed spacetime $M \in \mathsf{Loc}_{\mathrm{Ric}}$ is performed by the CCR functor $\mathfrak{CCR} : \mathsf{PoCh}_{\mathbb{R}} \to \mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}$, as per Equation (4.6). In order to give a homotopy AQFT on C, being this either $\mathsf{Loc}_{\mathrm{Ric}}$ or the slice category $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$, we need to introduce a suitable naturality condition for the retarded/advanced trivializations of Definition 3.3.12. In Definition 3.3.20 unshifted Poisson structures τ are built in terms of compatible pairs of retarded/advanced trivializations and a rule to perform the choice of such trivializations for any $M \in \mathsf{C}$ in a natural way is now requested.

Definition 4.2.9. Let C be either $\mathsf{Loc}_{\mathrm{Ric}}$ or $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$, for $\overline{M} \in \mathsf{Loc}_{\mathrm{Ric}}$. A C-*natural retarded/advanced trivialization* is a family

$$\Lambda^{\pm} := \{\Lambda^{\pm}_M \in \underline{\mathrm{hom}}\left(\mathfrak{Obs}_{\mathrm{pc/fc}}(M), \mathfrak{Obs}_{\mathrm{pc/fc}}(M)\right)_1\}_{M \in \mathsf{C}} \tag{4.33}$$

of retarded/advanced trivializations for each object $M \in \mathsf{C}$, such that

$$f^*(j_{\mathrm{pc/fc}} \Lambda^{\pm}_N \iota) f_* = j_{\mathrm{pc/fc}} \Lambda^{\pm}_M \iota \tag{4.34}$$

for each C-morphism $f : M \to N$. We recall that the maps $j_{\mathrm{pc/fc}}$ and ι have been introduced in Equation (3.85).

Remark 4.2.10. Note that the forgetful functor $\mathfrak{U}_{\overline{M}} : \mathsf{Loc}_{\mathrm{Ric}}/\overline{M} \to \mathsf{Loc}_{\mathrm{Ric}}$ of Remark 4.2.4 allows us to restrict any $\mathsf{Loc}_{\mathrm{Ric}}$-natural retarded/advanced trivialization to a $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$-natural one, for each $\overline{M} \in \mathsf{Loc}_{\mathrm{Ric}}$. This means that, in general, it is easier to construct natural retarded/advanced trivializations on the slice category $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$ than on $\mathsf{Loc}_{\mathrm{Ric}}$. ▽

We are now interested in proving that the retarded/advanced trivializations that we wrote explicitly in Equation (3.87) arrange themselves in a $\mathsf{Loc}_{\mathrm{Ric}}$-natural family.

Proposition 4.2.11. *The retarded/advanced trivializations for the linearized gravity theory of Proposition 3.3.14 define a $\mathsf{Loc}_{\mathrm{Ric}}$-natural retarded/advanced trivialization.*

Proof. We write here for simplicity the explicit form of the retarded/advanced trivializations of Proposition 3.3.14 for an object $M \in \mathsf{Loc}_{\mathrm{Ric}}$:

$$\Lambda^{\pm}_{M,-1} = -2G^M_{\pm}\nabla^M_S\,, \quad \Lambda^{\pm}_{M,0} = G^M_{\pm}\,, \quad \Lambda^{\pm}_{M,1} = 2\,\mathrm{div}^M\, G^M_{\pm}\,, \tag{4.35}$$

where $G^M_{\pm} : \Gamma_{pc/fc}(\otimes^2_S T^*M) \to \Gamma_{pc/fc}(\otimes^2_S T^*M)$ is the retarded/advanced Green operator for the differential operator $P'^M = \big(-\Box^M + 2\,\mathrm{Riem}\,(g_M)\big)\, I^M$. Observe that P' satisfies the naturality condition $f^*P'^N f_* = P'^M$ for all $\mathsf{Loc}_{\mathrm{Ric}}$-morphisms $f : M \to N$. Since pullbacks and pushforwards of $\mathsf{Loc}_{\mathrm{Ric}}$-morphisms do preserve the support properties of Green operators, the uniqueness of retarded/advanced Green operators for the Green hyperbolic operator P'^M imply that the naturality condition

$$f^* G^N_{\pm} f_* = G^M_{\pm} \tag{4.36}$$

holds true for all $f : M \to N$ morphisms in $\mathsf{Loc}_{\mathrm{Ric}}$. The reader is referred to [BG11] for further details. This naturality condition, together with those for the Killing operator and for the divergence operator (obtained via duality), implies that the identities in Equations (4.35) define a $\mathsf{Loc}_{\mathrm{Ric}}$-natural retarded/advanced trivialization. □

Let us give a naturality axiom for unshifted Poisson structures. It is a general condition given independently on the particular choice of the pair of C-natural retarded/advanced trivializations.

Definition 4.2.12. Let C be either $\mathsf{Loc}_{\mathrm{Ric}}$ or $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$, for any $\overline{M} \in \mathsf{Loc}_{\mathrm{Ric}}$. A C-*natural unshifted Poisson structure* is a family

$$\tau := \{\tau^M \in \underline{\mathrm{hom}}\,(\textstyle\bigwedge^2 \mathfrak{Obs}(M), \mathbb{R})_0\}_{M\in\mathsf{C}} \tag{4.37}$$

of unshifted Poisson structures for each $M \in \mathsf{C}$, such that the naturality condition, encoded in the diagram

$$\begin{array}{ccc} \mathfrak{Obs}(M) \wedge \mathfrak{Obs}(M) & \xrightarrow{\tau^M} & \mathbb{R} \\ {\scriptstyle f_*\wedge f_*}\downarrow & & \parallel \\ \mathfrak{Obs}(N) \wedge \mathfrak{Obs}(N) & \xrightarrow[\tau^N]{} & \mathbb{R} \end{array} \tag{4.38}$$

commutes for all C-morphisms $f : M \to N$. A C-*natural homotopy* between two C-natural unshifted Poisson structures τ and $\widetilde{\tau}$ is a family

$$\rho := \{\rho^M \in \underline{\hom}\,(\textstyle\bigwedge^2 \mathfrak{Obs}(M), \mathbb{R})_1\}_{M \in \mathsf{C}} \tag{4.39}$$

of 1-chains such that $\widetilde{\tau}^M - \tau^M = \partial \rho^M$, for all $M \in \mathsf{C}$, and $\rho^N(f_* \wedge f_*) = \rho^M$, for all C-morphisms $f : M \to N$.

Remark 4.2.13. Similarly to Remark 4.2.10, we observe that the $\mathsf{Loc}_{\mathrm{Ric}}$-natural unshifted Poisson structures and homotopies are harder to construct than the $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$-natural ones since all $\mathsf{Loc}_{\mathrm{Ric}}$-natural structures can be restricted to the $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$-natural ones via the precomposition with the forgetful functor $\mathfrak{U}_{\overline{M}} : \mathsf{Loc}_{\mathrm{Ric}}/\overline{M} \to \mathsf{Loc}_{\mathrm{Ric}}$. ▽

Now we want to highlight the link between C-natural retarded/advanced trivializations and C-natural unshifted Poisson structures. We are going to prove that C-natural retarded/advanced trivializations identify C-natural unshifted Poisson structures. Furthermore, we shall observe that when $\mathsf{C} = \mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$, for $\overline{M} \in \mathsf{Loc}_{\mathrm{Ric}}$, the $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$-natural unshifted Poisson structure which is built from $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$-natural retarded/advanced trivializations is unique up to homotopy. This will allow us to state a uniqueness result up to homotopy for the linearized gravity homotopy AQFT on $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$.

Lemma 4.2.14. *The following statements hold true:*

i. Let C be either $\mathsf{Loc}_{\mathrm{Ric}}$ or $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$, for any $\overline{M} \in \mathsf{Loc}_{\mathrm{Ric}}$. Suppose $\Lambda^{\pm}$ be a C-natural compatible pair of retarded/advanced trivializations. Then the component-wise construction of an unshifted Poisson structure according to Definition 3.3.20 defines a C-natural unshifted Poisson structure τ;

ii. Let $\mathsf{C} = \mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$, for any $\overline{M} \in \mathsf{Loc}_{\mathrm{Ric}}$. Then every $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$-natural unshifted Poisson structure τ is uniquely determined by an unshifted Poisson structure $\tau^{\overline{M}}$ on $\overline{M}$ and every $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$-natural homotopy ρ is uniquely determined by a homotopy $\rho^{\overline{M}}$ on $\overline{M}$;

iii. Let $\Lambda^{\pm}$ and $\widetilde{\Lambda}^{\pm}$ be two $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$-natural compatible pairs of retarded/advanced trivializations, for any $\overline{M} \in \mathsf{Loc}_{\mathrm{Ric}}$, then the corresponding $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$-natural unshifted Poisson structures $\tau, \widetilde{\tau}$ from item ii. are homotopic, i.e. *there exists a $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$-natural homotopy ρ such that $\widetilde{\tau} - \tau = \partial \rho$.*

Proof. Item i. follows immediately. Since $\Lambda^{\pm}$ is a C-natural compatible pair of retarded/advanced trivializations, the definition of the unshifted Poisson structure involves only natural maps and therefore it is natural too. Concerning items ii. and iii., it is helpful to observe that Definition 4.2.12 can be rephrased in terms of a categorical limit. Recall that $\mathfrak{Obs} : \mathsf{C} \to \mathsf{Ch}_{\mathbb{R}}$ is the functor that assigns chain complexes of linear observables. Then consider the mapping complex

$$\underline{\hom}\,(\textstyle\bigwedge^2 \mathfrak{Obs}, \mathbb{R}) := \lim\limits_{M \in \mathsf{C}^{\mathrm{op}}} \underline{\hom}\,(\textstyle\bigwedge^2 \mathfrak{Obs}(M), \mathbb{R}) \in \mathsf{Ch}_{\mathbb{R}}\,. \tag{4.40}$$

We note that a C-natural unshifted Poisson structure is a 0-cycle $\tau \in \underline{\mathrm{hom}}\,(\bigwedge^2 \mathfrak{Obs}, \mathbb{R})_0$, while a C-natural homotopy between two C-natural unshifted Poisson structures $\tau, \widetilde{\tau} \in \underline{\mathrm{hom}}\,(\bigwedge^2 \mathfrak{Obs}, \mathbb{R})_0$ is a 1-chain $\rho \in \underline{\mathrm{hom}}\,(\bigwedge^2 \mathfrak{Obs}, \mathbb{R})_1$, such that $\widetilde{\tau} - \tau = \partial\rho$. Let us assume $\mathsf{C} = \mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$. Then the slice category admits a terminal object, namely $(\mathrm{id} : \overline{M} \to \overline{M})$. Hence, the opposite category $(\mathsf{Loc}_{\mathrm{Ric}}/\overline{M})^{\mathrm{op}}$ has an initial object. The limit in Equation (4.40) is then isomorphic to the mapping complex $\underline{\mathrm{hom}}\,(\bigwedge^2 \mathfrak{Obs}(\overline{M}), \mathbb{R})$. This proves item ii.. Finally item iii. is a direct consequence of item ii. and of Lemma 3.3.25. □

Remark 4.2.15. Let us take a moment to comment this lemma, in particular item iii.. If we consider a theory on $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$, for a $\overline{M} \in \mathsf{Loc}_{\mathrm{Ric}}$, namely we are interested in a theory on a fixed spacetime $\overline{M}$, Lemma 4.2.14 guarantees us that our construction will provide a unique up to homotopy unshifted Poisson structure. Indeed, item iii. states that any possible choice of a $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$-natural compatible pair of retarded/advanced trivializations induces a $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$-natural unshifted Poisson structure belonging to the same homotopy class. A similar result may not hold true on the category $\mathsf{Loc}_{\mathrm{Ric}}$. Suppose that $\Lambda^{\pm}, \widetilde{\Lambda}^{\pm}$ are $\mathsf{Loc}_{\mathrm{Ric}}$-natural compatible pairs of retarded/advanced trivializations. Denote by $\tau, \widetilde{\tau}$ the corresponding $\mathsf{Loc}_{\mathrm{Ric}}$-natural unshifted Poisson structures. Whenever the spacetime $M \in \mathsf{Loc}_{\mathrm{Ric}}$ is fixed, Lemma 3.3.25 provides us with a 1-chain $\rho^M \in \underline{\mathrm{hom}}\,(\bigwedge^2 \mathfrak{Obs}(M), \mathbb{R})_1$ such that $\widetilde{\tau}^M - \tau^M = \partial\rho^M$. Nevertheless, it is not clear if those homotopies can always be chosen to be $\mathsf{Loc}_{\mathrm{Ric}}$-natural. In other words, it may be possible that different $\mathsf{Loc}_{\mathrm{Ric}}$-natural compatible pairs of retarded/advanced trivializations identify non-homotopic $\mathsf{Loc}_{\mathrm{Ric}}$-natural unshifted Poisson structures. These in turn may lead to non-equivalent quantum field theories. ▽

Remark 4.2.16. Consider the $\mathsf{Loc}_{\mathrm{Ric}}$-natural compatible pair of retarded/advanced trivializations from Proposition 4.2.11, whose components are written explicitly in Equations (4.35). It defines via Lemma 4.2.14, item i., a $\mathsf{Loc}_{\mathrm{Ric}}$-natural unshifted Poisson structure τ_{LG}, whose components τ_{LG}^M, for $M \in \mathsf{Loc}_{\mathrm{Ric}}$, are given by Equations (3.100). ▽

Let us explain how to build the quantum field theory functor for our setting. Let C be either the category $\mathsf{Loc}_{\mathrm{Ric}}$ or $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$, for any $\overline{M} \in \mathsf{Loc}_{\mathrm{Ric}}$, then suppose to select a C-natural unshifted Poisson structure τ. Below we will choose the Poisson structure τ_{LG} of Remark 4.2.16 as the C-natural unshifted Poisson structure for our particular model for quantum linearized gravity theory. The assignment $M \mapsto (\mathfrak{Obs}(M), \tau_M)$ of Poisson chain complexes defines a functor

$$(\mathfrak{Obs}, \tau) : \mathsf{C} \longrightarrow \mathsf{PoCh}_{\mathbb{R}}\,, \tag{4.41}$$

whose action on C-morphisms $f : M \to N$ is given by pushforwards $f_* : \mathfrak{Obs}(M) \to \mathfrak{Obs}(N)$. Observe that they define $\mathsf{PoCh}_{\mathbb{R}}$-morphisms due to the naturality condition on C as per Equation (4.38). The quantization is then achieved by post-composing with the CCR quantization functor as per Equation (4.6). We get a functor

$$\mathfrak{A} := \mathfrak{CCR} \circ (\mathfrak{Obs}, \tau) : \mathsf{C} \longrightarrow \mathsf{dg^*Alg}_{\mathbb{C}} \tag{4.42}$$

to the category of differential graded (unital) $*$-algebras.

Before proving that the choice of the particular C-natural unshifted Poisson structure τ_{LG} makes the corresponding functor $\mathfrak{A}_{\mathrm{LG}} = \mathfrak{CCR}(\mathfrak{Obs}, \tau_{\mathrm{LG}})$ into a homotopy AQFT in the sense of Definition 4.2.2 or Remark 4.2.3, we need to consider the homotopical properties of this construction. Recall that we endowed $\mathsf{PoCh}_{\mathbb{R}}$ and $\mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}$ with the structure of a homotopical category by declaring the weak equivalences of Definition 4.1.11. In order to analyze the homotopical properties of our quantization we induce a homotopical structure on both the functor categories $\mathsf{PoCh}_{\mathbb{R}}^{\mathsf{C}}$ and $\mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}^{\mathsf{C}}$. These are the categories whose objects are functors from C to $\mathsf{PoCh}_{\mathbb{R}}$ and $\mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}$, respectively, while morphisms are natural transformations between those functors.

Definition 4.2.17. Let C be either $\mathsf{Loc}_{\mathrm{Ric}}$ or $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$, for any $\overline{M} \in \mathsf{Loc}_{\mathrm{Ric}}$.

i. A morphism in $\mathsf{PoCh}_{\mathbb{R}}^{\mathsf{C}}$ is a *natural weak equivalence* if all its components are weak equivalences in $\mathsf{PoCh}_{\mathbb{R}}$;

ii. A morphism in $\mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}^{\mathsf{C}}$ is a *natural weak equivalence* if all its components are weak equivalences in $\mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}$.

Remark 4.2.18. Note that as a consequence of Definition 4.2.2 and of the subsequent Remark 4.2.3 we can introduce the category of homotopy AQFTs on C, $\mathsf{hAQFT}(\mathsf{C}) \subseteq \mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}^{\mathsf{C}}$, as the full subcategory of functors satisfying the homotopy AQFT axioms of Definition 4.2.2. We can endow $\mathsf{hAQFT}(\mathsf{C})$ with the structure of a homotopy category, with weak equivalences inherited from those of $\mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}^{\mathsf{C}}$. Hence, we declare a morphism in $\mathsf{hAQFT}(\mathsf{C})$ a weak equivalence if and only if it is a natural weak equivalence of $\mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}^{\mathsf{C}}$. ▽

We can now extend the results of Proposition 4.1.13 to the context of functor categories.

Proposition 4.2.19. *Let* C *be either* $\mathsf{Loc}_{\mathrm{Ric}}$ *or* $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$*, for any* $\overline{M} \in \mathsf{Loc}_{\mathrm{Ric}}$*. Then the post-composition with the CCR functor defines a homotopical functor*

$$\mathfrak{CCR} \circ (-) : \mathsf{PoCh}_{\mathbb{R}}^{\mathsf{C}} \longrightarrow \mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}^{\mathsf{C}}\,, \tag{4.43}$$

where domain and target categories are endowed with the natural weak equivalences of Definition 4.2.17. Moreover, let $(V,\tau) \in \mathsf{PoCh}_{\mathbb{R}}^{\mathsf{C}}$ *and* $\rho \in \underline{\mathrm{hom}}\,(\bigwedge^2 V, \mathbb{R})_1$ *be a* C*-natural* 1*-chain. Then, there exists a zigzag*

$$\mathfrak{CCR}(V,\tau) \stackrel{\sim}{\longleftarrow} A_{(V,\tau,\rho)} \stackrel{\sim}{\longrightarrow} \mathfrak{CCR}(V,\tau+\partial\rho) \tag{4.44}$$

of natural weak equivalences in $\mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}^{\mathsf{C}}$.

Proof. Let us start by proving that $\mathfrak{CCR} \circ (-)$ is homotopical. We need to check that it sends natural weak equivalences of $\mathsf{PoCh}_{\mathbb{R}}^{\mathsf{C}}$ to natural weak equivalences of $\mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}^{\mathsf{C}}$. Since the homotopical structures of both $\mathsf{PoCh}_{\mathbb{R}}^{\mathsf{C}}$ and $\mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}^{\mathsf{C}}$ are defined componentwise, this is a direct consequence of the first part of Proposition 4.1.13.

Let $M \in \mathsf{C}$, then the second part of Proposition 4.1.13 provides a zigzag

$$\mathfrak{CCR}(V(M), \tau^M) \xleftarrow{\sim} A_{(V(M),\tau^M,\rho^M)} \xrightarrow{\sim} \mathfrak{CCR}(V(M), \tau^M + \partial\rho^M) \tag{4.45}$$

of weak equivalences in $\mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}$. All we have to check is that those are components of natural transformations. We know from the proof of Proposition 4.1.13 that the zigzag (4.45) is obtained by applying the functor $\mathfrak{Q}_{\mathrm{lin}} : \mathsf{dg}^*\mathsf{uLie}_{\mathbb{C}} \to \mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}$ to the zigzag

$$\mathfrak{Heis}(V(M), \tau^M) \xleftarrow{\pi_0} H_{(V(M),\tau^M,\rho^M)} \xrightarrow{\pi_1} \mathfrak{Heis}(V(M), \tau^M + \partial\rho^M) \tag{4.46}$$

of weak equivalences in $\mathsf{dg}^*\mathsf{uLie}_{\mathbb{C}}$. The explicit expression for the object $H_{(V(M),\tau^M,\rho^M)}$ is written in the proof of Proposition 4.1.13. Let $f : M \to N$ be a C-morphism, then it follows that the diagram

$$\begin{array}{ccc} H_{(V(M),\tau^M,\rho^M)} & \xrightarrow{V(f)_{\mathbb{C}}\oplus\mathrm{id}\oplus\mathrm{id}} & H_{(V(N),\tau^N,\rho^N)} \\ {\scriptstyle \pi_s}\big\downarrow & & \big\downarrow{\scriptstyle \pi_s} \\ \mathfrak{Heis}(V(M), \tau^M + s\partial\rho^M) & \xrightarrow[V(f)_{\mathbb{C}}\oplus\mathrm{id}]{} & \mathfrak{Heis}(V(N), \tau^N + s\partial\rho^N) \end{array} \tag{4.47}$$

commutes for every $s \in \mathbb{R}$. By taking $s = 0$ and $s = 1$, one realizes that the arrows in Equation (4.46) are components of natural transformations. This concludes the proof. □

Corollary 4.2.20. *Suppose $\Lambda^{\pm}, \widetilde{\Lambda}^{\pm}$ are two $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$-natural compatible pairs of retarded/advanced trivializations, for a fixed $\overline{M} \in \mathsf{Loc}_{\mathrm{Ric}}$. Denote the corresponding $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$-natural unshifted Poisson structures from Lemma 4.2.14, item i., by τ and $\widetilde{\tau}$. Then the two functors $\mathfrak{A} := \mathfrak{CCR}(\mathfrak{Obs}, \tau) \simeq \mathfrak{CCR}(\mathfrak{Obs}, \widetilde{\tau}) =: \widetilde{\mathfrak{A}}$ are weakly equivalent via a zigzag of natural weak equivalences in $\mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}^{\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}}$.*

Proof. Lemma 4.2.14, item iii., guarantees that the two $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$-natural unshifted Poisson structures τ and $\widetilde{\tau}$ are homotopic. Then there exists a $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$-natural 1-chain $\rho \in \underline{\hom}\,(\bigwedge^2 \mathfrak{Obs}, \mathbb{R})_1$, such that $\widetilde{\tau} = \tau + \partial\rho$. Finally, Proposition 4.2.19 provides the sought zigzag of weak equivalences in $\mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}^{\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}}$. □

We need to prove that our particular prescription yields a homotopy AQFT. In other words, let C be either $\mathsf{Loc}_{\mathrm{Ric}}$ or $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$, for any $\overline{M} \in \mathsf{Loc}_{\mathrm{Ric}}$, and τ_{LG} be the C-natural unshifted Poisson structure defined by Lemma 4.2.14 item i. with respect to the C-natural compatible pair of retarded/advanced trivializations for linearized gravity theory of Proposition 4.2.11. Then, consider the functor

$$\mathfrak{A}_{\mathrm{LG}} := \mathfrak{CCR} \circ (\mathfrak{Obs}, \tau_{\mathrm{LG}}) : \mathsf{C} \longrightarrow \mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}} \tag{4.48}$$

which assigns the CCR algebra of quantum observables for linearized gravity to each spacetime $M \in \mathsf{C}$. We have to check that $\mathfrak{A}_{\mathrm{LG}} \in \mathsf{hAQFT}(\mathsf{C})$, namely that it satisfies Einstein causality axiom and the time-slice axiom, *cf.* Definition 4.2.2.

In order to do it we shall follow the same strategy of [BBS19]. This consists in checking suitable, easier to prove, conditions on the functor $(\mathfrak{Obs}, \tau_{\mathrm{LG}}) : \mathsf{C} \to \mathsf{PoCh}_{\mathbb{R}}$ which imply the aforementioned hAQFT axioms. The next lemma goes in this direction.

Lemma 4.2.21. *Let* C *be either* $\mathsf{Loc}_{\mathrm{Ric}}$ *or* $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$*, for any* $\overline{M} \in \mathsf{Loc}_{\mathrm{Ric}}$*, and consider a functor* $(V, \tau) : \mathsf{C} \to \mathsf{PoCh}_{\mathbb{R}}$*. Then,*

i. If for every pair $(f_1 : M_1 \to N, f_2 : M_2 \to N)$ *of* C*-morphisms with causally disjoint images the chain map*

$$\tau(f_{1*} \otimes f_{2*}) : V(M_1) \otimes V(M_2) \longrightarrow V(N) \tag{4.49}$$

vanishes, then the functor $\mathfrak{A} := \mathfrak{CCR} \circ (V, \tau) : \mathsf{C} \to \mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}$ *satisfies Einstein causality;*

ii. If for every Cauchy morphism $f : M \to N$ *the chain map* $f_* : V(M) \to V(N)$ *is a quasi-isomorphism, then functor* $\mathfrak{A} := \mathfrak{CCR} \circ (V, \tau) : \mathsf{C} \to \mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}$ *satisfies the time-slice axiom.*

Proof. Item i. follows directly from the explicit form of the canonical commutation relations in Equation (4.3). In fact, it is immediate to realize that

$$[\mathfrak{A}(f_1)v_1, \mathfrak{A}(f_2)v_2] = i\tau(f_{1*}v_1 \otimes f_{2*}v_2)\mathbb{1} = 0\,, \tag{4.50}$$

for any homogeneous elements $v_1 \in V(M_1)_n$ and $v_2 \in V(M_2)_m$, for $n, m \in \mathbb{Z}$. The time-slice axiom requires that, for any Cauchy morphism $f : M \to N$, the map $\mathfrak{A}(f) : \mathfrak{A}(M) \to \mathfrak{A}(N)$ is a weak equivalence in $\mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}$. Under the hypotheses of item ii. the $\mathsf{PoCh}_{\mathbb{R}}$-morphism $(V, \tau)(f)$ is a weak equivalence in $\mathsf{PoCh}_{\mathbb{R}}$ in the sense of Definition 4.1.11, item i.. Since the CCR functor is homotopical, see Proposition 4.1.13, the map $\mathfrak{CCR}(V, \tau)(f)$ is a weak equivalence in $\mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}$ and therefore the time-slice axiom is fulfilled. □

We start by proving some preliminary lemmas that we are going to use to prove that $\mathfrak{A}_{\mathrm{LG}}$ satisfies the time-slice axiom.

Lemma 4.2.22. *Let* (N, g) *be a globally hyperbolic Ricci-flat manifold and* $\widetilde{M} \subseteq N$ *a causally convex open submanifold of* N *which contains a Cauchy surface of* N*. Let us consider on* $\widetilde{M}$ *the metric* $g|_{\widetilde{M}}$ *given by the restriction of* g*. Then, any Killing field* $\chi \in \Gamma(T^*\widetilde{M})$ *on the submanifold* $\widetilde{M}$ *admits an extension to a Killing field* $\widetilde{\chi} \in \Gamma(T^*N)$ *on the entire manifold* N*.*

Proof. Let $\chi \in \Gamma(T^*\widetilde{M})$ be any Killing field on the submanifold $\widetilde{M}$. Hence, χ is in the kernel of the Killing operator $\nabla_S : \Gamma(T^*\widetilde{M}) \to \Gamma(\otimes^2_S T^*\widetilde{M})$. By taking the contraction with the metric of the identity $\nabla_S\chi = 0$ we get $\nabla^a\chi_a = 0$. Then, we calculate

$$0 = 2\,\mathrm{div}\,\nabla_S\chi_a = \nabla^b(\nabla_b\chi_a + \nabla_a\chi_b) = \Box\chi_a + \nabla_a\nabla^b\chi_b = \Box\chi_a\,, \tag{4.51}$$

where we used Ricci-flatness to commute the covariant derivatives. Since χ is a solution of the hyperbolic equation (4.51) on a globally hyperbolic manifold, it admits a unique extension $\widetilde{\chi} \in \Gamma(T^*N)$ such that $\widetilde{\chi}|_{\widetilde{M}} = \chi$ and $\Box\widetilde{\chi} = 0$. Furthermore,

$$0 = \nabla_S \Box \widetilde{\chi} = (\Box - 2\,\mathrm{Riem})\nabla_S \widetilde{\chi}\,, \tag{4.52}$$

where the second step follows from Lemma 2.1.13. Since $(\Box - 2\,\mathrm{Riem})$ is a normally hyperbolic operator and $\nabla_S\widetilde{\chi} = 0$ in $\widetilde{M}$ it follows that $\nabla_S\widetilde{\chi}$ is everywhere vanishing in N. Hence, $\widetilde{\chi} \in \mathrm{Ker}\,\nabla_S \subseteq \Gamma(T^*N)$ and thus it is a Killing field on N. $\Box$

We recall that, according to our notation, Ker_c (Im_c) denotes the kernel (respectively, the image) of operators restricted to compactly supported sections.

Lemma 4.2.23. *Let N be a globally hyperbolic Ricci-flat manifold and $\widetilde{M} \subseteq N$ a causally convex open submanifold of N which contains a Cauchy surface of N. Then, the map*

$$\begin{aligned} \iota : \Gamma_c(T^*\widetilde{M})/\,\mathrm{Im}_c\,(\mathrm{div}^{\widetilde{M}}) &\longrightarrow \Gamma_c(T^*N)/\,\mathrm{Im}_c\,(\mathrm{div}^{N}) \\ [\zeta] &\longmapsto \iota[\zeta] \coloneqq [\overline{\zeta}]\,, \end{aligned} \tag{4.53}$$

where $\overline{\zeta} \in \Gamma_c(T^*N)$ denotes the extension of $\zeta \in \Gamma_c(T^*\widetilde{M})$ which is obtained by defining it equals to zero on $N \setminus \widetilde{M}$, is an isomorphism.

Proof. Let us start with a geometric construction that will be useful also in the following proofs. Let $\Sigma \subseteq N$ be a Cauchy surface entirely contained in $\widetilde{M}$. Consider two other Cauchy surfaces, $\Sigma_\pm \subseteq \widetilde{M}$, in the future and in the past of Σ, respectively. Then, take the cover of N given by $I_\mp(\Sigma_\pm)$, where $I_\mp(-)$ denotes the chronological past/future of a subset. Note that $I_-(\Sigma_+) \cap I_+(\Sigma_-) \subseteq \widetilde{M}$ because of the causal convexity of $\widetilde{M}$. Let us introduce a partition of unity $\{\chi_+, \chi_-\}$ subordinate to this cover, namely, smooth scalar functions $\chi_\pm \in C^\infty(N)$, such that $\mathrm{supp}\,\chi_\pm \subset I_\pm(\Sigma_\mp)$ and $\chi_+ + \chi_- = 1$ on N.

In order to prove the surjectivity, let $[\eta] \in \Gamma_c(T^*N)/\,\mathrm{Im}_c\,(\mathrm{div}^N)$ be any equivalence class and let η be an arbitrary representative. Let us set

$$\widetilde{\eta} \coloneqq \Box\chi_+ G^{\Box}\eta \in \Gamma(T^*N)\,, \tag{4.54}$$

where $G^{\Box} : \Gamma_c(T^*N) \to \Gamma_{sc}(T^*N)$ is the causal propagator for $\Box = \nabla^a\nabla_a$. The identity

$$\widetilde{\eta} = \Box(1 - \chi_-)G^{\Box}\eta = -\Box\chi_- G^{\Box}\eta\,, \tag{4.55}$$

together with Equation (4.54), allows us to conclude that $\mathrm{supp}\,\widetilde{\eta} \subseteq \mathrm{supp}\,\chi_+ \cap \mathrm{supp}\,\chi_- \cap J(\mathrm{supp}\,\eta)$, hence $\widetilde{\eta} \in \Gamma_c(T^*N)$ has compact support in $\widetilde{M}$. Consequently, the restriction of $\widetilde{\eta}$ to $\widetilde{M}$ identifies uniquely an equivalence class $[\widetilde{\eta}] \in \Gamma_c(T^*\widetilde{M})/\,\mathrm{Im}_c\,(\mathrm{div}^{\widetilde{M}})$. Observe that $[\widetilde{\eta}] = [\eta]$. In fact,

$$\begin{aligned} \widetilde{\eta} - \eta &= \Box\chi_+ G^{\Box}\eta - \Box G^{\Box}_+\eta \\ &= \Box\chi_+(G^{\Box}_+ - G^{\Box}_-)\eta - \Box(\chi_+ + \chi_-)G^{\Box}_+\eta \\ &= \Box(-\chi_+ G^{\Box}_- - \chi_- G^{\Box}_+)\eta \\ &= \mathrm{div}\,\{-2I\nabla_S(\chi_+ G^{\Box}_- + \chi_- G^{\Box}_+)\eta\}\,, \end{aligned} \tag{4.56}$$

where in the last step we used identity from Lemma 2.1.13. Equation (4.56) expresses the difference between $\widetilde{\eta}$ and η in the form of the divergence of $-2I\nabla_S(\chi_+G^\square_- + \chi_-G^\square_+)\eta$, which has compact support in N as a consequence of the support properties of the functions $\chi_\pm$ and of the retarded/advanced Green operators $G^\square_\pm$. This proves that ι is surjective.

Concerning injectivity, we consider an equivalence class $[\eta] \in \Gamma_c(T^*\widetilde{M})/\operatorname{Im}_c(\operatorname{div}^{\widetilde{M}})$ such that $\iota[\eta] = [0]$ in $\Gamma_c(T^*N)/\operatorname{Im}_c(\operatorname{div}^N)$. We have to prove that $[\eta] = [0]$ as an equivalence class in $\Gamma_c(T^*\widetilde{M})/\operatorname{Im}_c(\operatorname{div}^{\widetilde{M}})$. Let us consider the pairing (3.66) on the level of homologies:

$$(-,-) : \Gamma_c(T^*\widetilde{M})/\operatorname{Im}_c(\operatorname{div}^{\widetilde{M}}) \otimes \operatorname{Ker}(\nabla_S^{\widetilde{M}}) \longrightarrow \mathbb{R}\,, \quad ([\eta],\chi) \coloneqq \int_{\widetilde{M}} \eta_a \chi_b g^{ab} \mu_g\,, \tag{4.57}$$

where an arbitrary representative is picked in the equivalence class. Let $\chi \in \operatorname{Ker}(\nabla_S^{\widetilde{M}})$ be any Killing field and let $\widetilde{\chi} \in \operatorname{Ker}(\nabla_S^N)$ be its extension to N given by Lemma 4.2.22. Then, let us consider the pairing

$$([\eta],\chi) = (\iota[\eta],\widetilde{\chi}) = 0\,, \tag{4.58}$$

where we exploit the hypothesis that $\iota[\eta] = [0]$. Since $\chi \in \operatorname{Ker}(\nabla_S^{\widetilde{M}})$ is arbitrary and the pairing (4.58) is non-degenerate, it follows that $[\eta] = [0]$. □

Lemma 4.2.24. *Let N be a globally hyperbolic Ricci-flat manifold and $\widetilde{M} \subseteq N$ a causally convex open submanifold of N which contains a Cauchy surface of N. Then, the map*

$$\begin{aligned} \iota' : \operatorname{Ker}_c P^{\widetilde{M}}/\operatorname{Im}_c \nabla_S^{\widetilde{M}} &\longrightarrow \operatorname{Ker}_c P^N/\operatorname{Im}_c \nabla_S^N \\ [\omega] &\longmapsto \iota'[\omega] \coloneqq [\overline{\omega}]\,, \end{aligned} \tag{4.59}$$

where $\overline{\omega} \in \Gamma_c(\otimes_S^2 T^*N)$ denotes the extension of $\omega \in \Gamma_c(\otimes_S^2 T^*\widetilde{M})$ which is obtained by defining it equals to zero on $N \setminus \widetilde{M}$, is an isomorphism.

Proof. First, we introduce a partition of unity $\{\chi_+,\chi_-\}$ as per the construction at the beginning of the proof of Lemma 4.2.23.

Let us consider the surjectivity. Let $[\alpha] \in \operatorname{Ker}_c P^N/\operatorname{Im}_c \nabla_S^N$ be any equivalence class and let us pick an arbitrary representative $\alpha \in [\alpha]$. We want to show that

$$\widetilde{\alpha} \coloneqq -2\nabla_S \operatorname{div}\chi_+ G\alpha \in \Gamma(\otimes_S^2 T^*N) \tag{4.60}$$

identifies an equivalence class $[\widetilde{\alpha}] \in \operatorname{Ker}_c P^{\widetilde{M}}/\operatorname{Im}_c \nabla_S^{\widetilde{M}}$ such that $\iota'[\widetilde{\alpha}] = [\alpha]$. First, observe that $\alpha \in \operatorname{Ker}_c P^N$ implies the identity

$$P'\alpha = -2I\nabla_S \operatorname{div} I\alpha\,. \tag{4.61}$$

Hence,

$$\begin{aligned}\widetilde{\alpha} &= -2\nabla_S \mathrm{div}\,(1-\chi_-)G\alpha = -2\nabla_S \mathrm{div}\, IGI\alpha + 2\nabla_S \mathrm{div}\,\chi_- G\alpha \\ &= 2\nabla_S G^{\Box}\,\mathrm{div}\, I\alpha + 2\nabla_S \mathrm{div}\,\chi_- G\alpha = -2GI\nabla_S \mathrm{div}\, I\alpha + 2\nabla_S \mathrm{div}\,\chi_- G\alpha \\ &= GP'\alpha + 2\nabla_S \mathrm{div}\,\chi_- G\alpha = 2\nabla_S \mathrm{div}\,\chi_- G\alpha\,, \end{aligned} \tag{4.62}$$

where Lemma 2.2.5 and its dual are used, together with the identity (4.61) and the definition of the causal propagator G. The same reasonings as in the proof of Lemma 4.2.23 allow us to conclude that $\mathrm{supp}\,\widetilde{\alpha} \subseteq \mathrm{supp}\,\chi_+ \cap \mathrm{supp}\,\chi_- \cap J(\mathrm{supp}\,\alpha)$, hence $\widetilde{\alpha}|_{\widetilde{M}} \in \Gamma_c(\otimes^2_S T^*\widetilde{M})$. More precisely $\widetilde{\alpha}|_{\widetilde{M}} \in \mathrm{Ker}_c\, P^{\widetilde{M}}$, since

$$P^{\widetilde{M}}\widetilde{\alpha}|_{\widetilde{M}} = (P^N\widetilde{\alpha})|_{\widetilde{M}} = 0\,, \tag{4.63}$$

due to the identity $P\nabla_S = 0$ from Proposition 2.1.14. This is tantamount to saying that $[\widetilde{\alpha}]$ is a well defined equivalence class in $\mathrm{Ker}_c\, P^{\widetilde{M}} / \,\mathrm{Im}_c\, \nabla_S^{\widetilde{M}}$. Surjectivity follows if we show that $[\widetilde{\alpha}] \sim [\alpha]$. Let us calculate,

$$\begin{aligned}\widetilde{\alpha} - \alpha &= -2\nabla_S \mathrm{div}\,\chi_+ G\alpha - G_+P'\alpha = -2\nabla_S \mathrm{div}\,\chi_+ G\alpha + 2G_+ I\nabla_S \mathrm{div}\, I\alpha \\ &= -2\nabla_S \mathrm{div}\,\chi_+(G_+ - G_-)\alpha + 2\nabla_S \mathrm{div}\, IG_+I\alpha \\ &= -2\nabla_S \mathrm{div}\,\chi_+(G_+ - G_-)\alpha + 2\nabla_S \mathrm{div}\,(\chi_+ + \chi_-)G_+\alpha \\ &= \nabla_S\{2\,\mathrm{div}\,(\chi_+G_- + \chi_-G_+)\alpha\}\,, \end{aligned} \tag{4.64}$$

where Equation (4.61) is used in the second step and Lemma 2.2.5 and its dual in the third one. The section $2\,\mathrm{div}\,(\chi_+G_- + \chi_-G_+)\alpha$ has compact support as a consequence of the support properties of the retarded/advanced Green operators $G_\pm$ and the those of the partition of unity $\{\chi_+, \chi_-\}$.

As far as the injectivity is concerned, let $[\alpha] \in \mathrm{Ker}_c\, P^{\widetilde{M}} / \,\mathrm{Im}_c\, \nabla_S^{\widetilde{M}}$ be such that $\iota'[\alpha] = [0]$. Then, let $[\eta] \in \Gamma_c(T^*\widetilde{M}) / \,\mathrm{Im}_c\,(\mathrm{div}^{\widetilde{M}})$ be any equivalence class. It holds

$$\tau^{\widetilde{M}}_{\mathrm{LG}}([\eta], [\alpha]) = \tau^N_{\mathrm{LG}}([\overline{\eta}], [\overline{\alpha}]) = 0\,. \tag{4.65}$$

By exploiting Equation (4.65) and selecting arbitrary representatives in both the equivalence classes, we find

$$0 = -\int_{\widetilde{M}} \eta_a 2\,(\mathrm{div}\, G\alpha)_b\, g^{ab}\mu_g = -(\eta, 2\,\mathrm{div}\, G\alpha)\,, \tag{4.66}$$

for all $\eta \in \Gamma_c(T^*\widetilde{M})$. Thanks to the non-degeneracy of the integral pairing, Equation (4.66) implies that $\mathrm{div}\, G\alpha = 0$. Lemma 2.2.5 yields the identity $G^{\Box}\,\mathrm{div}\, I\alpha = 0$. Hence, there exists a compactly supported section $\gamma \in \Gamma_c(T^*\widetilde{M})$ such that $\mathrm{div}\, I\alpha = \Box\gamma$ as a consequence of the general properties of the causal propagator. The last identity of Lemma 2.1.13 allows us to write

$$\mathrm{div}\, I\,(\alpha - 2\nabla_S\gamma) = 0\,. \tag{4.67}$$

Observe that $\alpha - 2\nabla_S\gamma \in \Gamma_c(\otimes^2_S T^*\widetilde{M})$ has compact support since it is a sum of compactly supported sections. Moreover, it holds

$$P\,(\alpha - 2\nabla_S\gamma) = 0\,, \tag{4.68}$$

as a consequence of $\alpha \in \mathrm{Ker}_c\, P^{\widetilde{M}}$ and of Proposition 2.1.14. Equations (4.67) and (4.68) entail that $P'\,(\alpha - 2\nabla_S\gamma) = 0$ and this implies in turn

$$\alpha - 2\nabla_S\gamma = 0\,, \tag{4.69}$$

on account of the exact sequence (2.32). We conclude that each representative in the equivalence class $[\alpha]$ is in the image of $\nabla_S : \Gamma_c(T^*\widetilde{M}) \to \Gamma_c(\otimes^2_S T^*\widetilde{M})$, hence $[\alpha] = 0$ and the injectivity of ι' is proved. □

We are ready to prove the final result of this work which shows that our prescription yields a homotopy AQFT for linearized gravity theory.

Theorem 4.2.25. *Let τ_{LG} be the $\mathsf{Loc}_{\mathrm{Ric}}$-natural unshifted Poisson structure corresponding to the $\mathsf{Loc}_{\mathrm{Ric}}$-natural compatible pair of retarded/advanced trivializations for linearized gravity from Proposition 4.2.11. Then the functor $\mathfrak{A}_{\mathrm{LG}} := \mathfrak{CCR} \circ (\mathfrak{Obs}, \tau_{\mathrm{LG}}) : \mathsf{Loc}_{\mathrm{Ric}} \to \mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}$ is a homotopy AQFT on $\mathsf{Loc}_{\mathrm{Ric}}$,* i.e. $\mathfrak{A}_{\mathrm{LG}} \in \mathsf{hAQFT}(\mathsf{Loc}_{\mathrm{Ric}})$. *Furthermore, let $\overline{M} \in \mathsf{Loc}_{\mathrm{Ric}}$ be any spacetime. Then the restriction $\mathfrak{A}^{\overline{M}}_{\mathrm{LG}} := \mathfrak{A}_{\mathrm{LG}}\, \mathfrak{U}_{\overline{M}} \in \mathsf{hAQFT}(\mathsf{Loc}_{\mathrm{Ric}}/\overline{M})$ defines a homotopy AQFT on $\overline{M}$. These homotopy AQFTs on a fixed spacetime $\overline{M}$ are determined uniquely up to weak equivalences.*

Proof. Lemma 4.2.14, item i., allows us to define a functor $(\mathfrak{Obs}, \tau_{\mathrm{LG}}) : \mathsf{Loc}_{\mathrm{Ric}} \to \mathsf{PoCh}_{\mathbb{R}}$, where $\mathfrak{Obs} : \mathsf{Loc}_{\mathrm{Ric}} \to \mathsf{Ch}_{\mathbb{R}}$ is the functor of Equation (4.29) which assigns the chain complex of linear observables for linearized gravity. Hence, post-composition with the CCR functor yields a functor $\mathfrak{A}_{\mathrm{LG}} := \mathfrak{CCR}(\mathfrak{Obs}, \tau_{\mathrm{LG}}) : \mathsf{Loc}_{\mathrm{Ric}} \to \mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}$. We have to prove that this functor fulfills the homotopy AQFT axioms of Definition 4.2.2. For this purpose we exploit the sufficient conditions on $(\mathfrak{Obs}, \tau_{\mathrm{LG}})$ provided by Lemma 4.2.21.

Let us start with Einstein causality. The sufficient condition from Lemma 4.2.21 item i. is satisfied as a consequence of the explicit expression for τ_{LG} that one can read from Equations (3.100) and of support properties of retarded/advanced Green operators.

We need to check that the hypotheses of Lemma 4.2.21 item ii. are also fulfilled, thus implying that $\mathfrak{A}_{\mathrm{LG}}$ satisfies the time-slice axiom. Let $f : M \to N$ be a Cauchy morphism and let us introduce the notation $\widetilde{M} := f(M) \subseteq N$. Observe that f factorizes through the subset inclusion $i : \widetilde{M} \to N$,

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & N \\ & {\scriptstyle \widetilde{f}}\searrow \quad \nearrow{\scriptstyle i} & \\ & \widetilde{M} & \end{array} \tag{4.70}$$

where $\widetilde{f} : M \to \widetilde{M}$ is an isometric diffeomorphism, hence an isomorphism in $\mathsf{Loc}_{\mathrm{Ric}}$. Because of functoriality $\widetilde{f}_* : \mathfrak{Obs}(M) \to \mathfrak{Obs}(\widetilde{M})$ is an isomorphism in $\mathsf{Ch}_{\mathbb{R}}$. Therefore, the time-slice axiom follows if we show that the canonical maps $H_n(i_*) : H_n(\mathfrak{Obs}(\widetilde{M})) \to H_n(\mathfrak{Obs}(N))$, $[\phi] \mapsto [\overline{\phi}]$, are isomorphisms. We recall that $\overline{\phi}$ denotes the extension to N of the compactly supported section ϕ which vanishes in $N \setminus \widetilde{M}$.

From our analysis in Section 3.3, we found that the only non-trivial homologies are those in degrees $n = -1, 0, 1$. Hence we restrict our attention to these cases. Since $H_{-1}(i_*) = \iota : \Gamma_c(T^*\widetilde{M}) / \operatorname{Im}_c(\operatorname{div}^{\widetilde{M}}) \to \Gamma_c(T^*N) / \operatorname{Im}_c(\operatorname{div}^N)$ and $H_1(i_*) = \iota' : \operatorname{Ker}_c P^{\widetilde{M}} / \operatorname{Im}_c \nabla_S^{\widetilde{M}} \to \operatorname{Ker}_c P^N / \operatorname{Im}_c \nabla_S^N$, the time-slice axiom is satisfied in degrees $n = -1$ and $n = 1$ as a consequence of Lemma 4.2.23 and of Lemma 4.2.24, respectively.

It remains to check it in degree $n = 0$. We have $H_0(i_*) : \operatorname{Ker}_c(\operatorname{div}^{\widetilde{M}}) / \operatorname{Im}_c(P^{\widetilde{M}}) \to \operatorname{Ker}_c(\operatorname{div}^N) / \operatorname{Im}_c(P^N)$. Observe that it corresponds to the usual pushforward along Cauchy morphisms of linear gauge invariant on-shell observables for linearized gravity. This is an isomorphism as shown in [FH13]. We sketch the proof reported there. Let us consider the partition of unity $\{\chi_+, \chi_-\}$ as per the construction at the beginning of the proof of Lemma 4.2.23. Starting with surjectivity, let $[\varepsilon] \in \operatorname{Ker}_c(\operatorname{div}^N) / \operatorname{Im}_c(P^N)$ be any equivalence class and let ε be an arbitrary representative. We define

$$\widetilde{\varepsilon} \coloneqq P\chi_+ G\varepsilon \in \Gamma_{sc}(\otimes_S^2 T^*N)\,, \tag{4.71}$$

where $G : \Gamma_c(\otimes_S^2 T^*N) \to \Gamma_{sc}(\otimes_S^2 T^*N)$ is the causal propagator for P', the linearized gravity operator in the de Donder gauge, see Equation (2.25a). It is immediate to see that $\operatorname{supp}\widetilde{\varepsilon} \subseteq \operatorname{supp}\chi_+$. Observe that $G\varepsilon$ is a solution of the linearized gravity equation of motion in the de Donder gauge. Indeed, it holds

$$\operatorname{div} IG\varepsilon = -G^{\Box} \operatorname{div}\varepsilon = 0\,, \tag{4.72}$$

where we used Lemma 2.2.5 and $\varepsilon \in \operatorname{Ker}_c(\operatorname{div}^N)$. Then, $PG\varepsilon = P'G\varepsilon = 0$. By exploiting the identity $\chi_+ = 1 - \chi_-$, we get

$$\widetilde{\varepsilon} = P(1-\chi_-)G\varepsilon = -P\chi_- G\varepsilon\,. \tag{4.73}$$

Hence, it also holds $\operatorname{supp}\widetilde{\varepsilon} \subseteq \operatorname{supp}\chi_-$ and consequently $\operatorname{supp}\widetilde{\varepsilon} \subseteq \operatorname{supp}\chi_+ \cap \operatorname{supp}\chi_- \cap J(\operatorname{supp}\varepsilon)$. Therefore, $\widetilde{\varepsilon}$ is compactly supported in $\widetilde{M}$ and its restriction $\widetilde{\varepsilon}|_{\widetilde{M}}$ identifies an element of $\Gamma_c(\otimes_S^2 T^*\widetilde{M})$. Moreover, $\widetilde{\varepsilon}|_{\widetilde{M}} \in \operatorname{Ker}_c(\operatorname{div}^{\widetilde{M}})$ as a direct consequence of gauge invariance. Hence, $\widetilde{\varepsilon}$ uniquely identifies an equivalence class $[\widetilde{\varepsilon}] \in H_0(\mathfrak{Obs}(\widetilde{M}))$. In order to complete the proof we need to check that $[\widetilde{\varepsilon}] = [\varepsilon]$ in $H_0(\mathfrak{Obs}(N))$. Then, we compute

$$\begin{aligned}
\widetilde{\varepsilon} - \varepsilon &= P\chi_+ G\varepsilon - PG_+\varepsilon \\
&= P\chi_+(G_+ - G_-)\varepsilon - P(\chi_+ + \chi_-)G_+\varepsilon \\
&= -P\chi_+ G_-\varepsilon - P\chi_- G_+\varepsilon \\
&= P(-\chi_+ G_- - \chi_- G_+)\varepsilon\,,
\end{aligned} \tag{4.74}$$

where in the first step we used $\operatorname{div} IG_+\varepsilon = 0$ and $P'G_+ = \mathrm{id}$, where id is the identity on $\Gamma_c(\otimes_S^2 T^*N)$. In the last step it appears the section $(-\chi_+ G_- - \chi_- G_+)\varepsilon$ which has compact support because of the general properties of the advanced/retarded Green operators and of the support properties of the partition of unity $\{\chi_+, \chi_-\}$. Hence $[\widetilde{\varepsilon}] \sim [\varepsilon]$ in $H_0(\mathfrak{Obs}(N))$.

As far as the injectivity is concerned, let $[\varepsilon] \in H_0(\mathfrak{Obs}(\widetilde{M}))$ such that $H_0(i_*)[\varepsilon] = [0]$ as an equivalence class in $H_0(\mathfrak{Obs}(N))$. This means that there exists $\alpha \in \Gamma_c(\otimes_S^2 T^*N)$ such that $\overline{\varepsilon} = P\alpha$. We exploit gauge invariance of the operator P in order to write α in the de Donder gauge. Then, for any $\beta \in \Gamma(T^*N)$ the identity $\overline{\varepsilon} = P(\alpha + \nabla_S \beta)$ holds true, as a consequence of Proposition 2.1.14. Let $\alpha' \coloneqq \alpha + \nabla_S \beta$. We want to find β such that $\operatorname{div} I\alpha' = 0$. This corresponds to solve equation

$$\Box \beta = -2 \operatorname{div} I\alpha \,. \tag{4.75}$$

Since N is globally hyperbolic such $\beta \in \Gamma_{sc}(T^*N)$ exists. This choice allows us to write $\overline{\varepsilon} = P'\alpha'$, where P' is the operator as per Equation (2.25a). By exploiting the retarded/advanced propagator $G_\pm$ for P', we get

$$\alpha' = G_\pm \overline{\varepsilon} \,. \tag{4.76}$$

The support properties of retarded/advanced propagators imply that

$$\operatorname{supp} \alpha' \subseteq J_-(\operatorname{supp} \overline{\varepsilon}) \cap J_+(\operatorname{supp} \overline{\varepsilon}) \subseteq \widetilde{M} \,, \tag{4.77}$$

hence, $\alpha' \in \Gamma_c(\otimes_S^2 T^*N)$ with $\operatorname{supp} \alpha' \subseteq \widetilde{M}$. We conclude that $\varepsilon = P'\alpha'|_{\widetilde{M}} = P\alpha'|_{\widetilde{M}}$, where the last step follows since $\operatorname{div} I\alpha' = 0$ by construction. Hence, $[\varepsilon] = [0]$ in $H_0(\mathfrak{Obs}(\widetilde{M}))$ and $H_0(i_*)$ is proved to be injective.

Summing up, this shows that $\mathfrak{A}_{\mathrm{LG}} : \mathsf{Loc}_{\mathrm{Ric}} \to \mathsf{dg}^*\mathsf{Alg}_{\mathbb{C}}$ satisfies the homotopy AQFT axioms, hence it is a homotopy AQFT on $\mathsf{Loc}_{\mathrm{Ric}}$. The restriction of the theory, $\mathfrak{A}_{\mathrm{LG}}^{\overline{M}} \coloneqq \mathfrak{A}_{\mathrm{LG}}\, \mathfrak{U}_{\overline{M}}$, on any $\overline{M} \in \mathsf{Loc}_{\mathrm{Ric}}$, identifies a homotopy AQFT on $\mathsf{Loc}_{\mathrm{Ric}}/\overline{M}$, as we have observed in Remark 4.2.3. The uniqueness (up to natural weak equivalences) of our construction for the restricted linearized gravity AQFTs, $\mathfrak{A}_{\mathrm{LG}}^{\overline{M}} \in \mathsf{hAQFT}(\mathsf{Loc}_{\mathrm{Ric}}/\overline{M})$, for each $\overline{M} \in \mathsf{Loc}_{\mathrm{Ric}}$, is a consequence of Corollary 4.2.20. □

Remark 4.2.26. Let us go back to the question of the uniqueness up to natural weak equivalences of our construction on $\mathsf{Loc}_{\mathrm{Ric}}$. The issue pointed out in Remark 4.2.15 extends to the context of AQFTs. It may be possible that there exists another $\mathsf{Loc}_{\mathrm{Ric}}$-natural compatible pair of retarded/advanced trivializations that identifies a $\mathsf{Loc}_{\mathrm{Ric}}$-natural unshifted Poisson structure non-homotopic to our τ_{LG}. Moreover, this could yield a model $\widetilde{\mathfrak{A}} \in \mathsf{hAQFT}(\mathsf{Loc}_{\mathrm{Ric}})$ for linearized gravity that is not equivalent to $\mathfrak{A}_{\mathrm{LG}}$. Nevertheless, Theorem 4.2.25 implies that for any $\overline{M} \in \mathsf{Loc}_{\mathrm{Ric}}$ the restricted theories $\widetilde{\mathfrak{A}}^{\overline{M}}, \mathfrak{A}_{\mathrm{LG}}^{\overline{M}} \in \mathsf{hAQFT}(\mathsf{Loc}_{\mathrm{Ric}}/\overline{M})$ are naturally weakly equivalent homotopy AQFTs on $\overline{M}$. In other words, if there exist non-equivalent quantum theories for linearized gravity on $\mathsf{Loc}_{\mathrm{Ric}}$, they will differ subtly since their restrictions to any fixed spacetime are equivalent. ▽

Remark 4.2.27. Note that our model for linearized gravity as a homotopy AQFT on $\mathsf{Loc}_{\mathrm{Ric}}$ is not always naturally weakly equivalent to models that consider only gauge-invariant on-shell observables, see *e.g.* [FH13; BDM14]. In fact, for an arbitrary background spacetime $M \in \mathsf{Loc}_{\mathrm{Ric}}$, the complex of linear observables $\mathfrak{Obs}(M)$, as per Definition 3.3.1, has non-trivial homology in degrees $n = -1, 0, 1$. This is in particular true for some non-simply connected physical spacetimes M of constant curvature, see the discussions in Remark 3.2.10, Example 3.2.12 and Remark 3.3.6. Models that take into account only gauge invariant on-shell observables consider only the zeroth homology of $\mathfrak{Obs}(M)$, ignoring information contained in the other ones. With our notation, the latter coincide with theories whose $*$-algebra of quantum observables is $\mathfrak{CCR}(H_0(\mathfrak{Obs}(M), \tau_{\mathrm{LG}}^M))$. This differs from our $\mathfrak{A}_{\mathrm{LG}}(M)$ even on the level of the zeroth homology. In fact, if the $*$-algebra $\mathfrak{CCR}(H_0(\mathfrak{Obs}(M), \tau_{\mathrm{LG}}^M))$ is generated only by linear gauge invariant on-shell observables, while the $*$-algebra $H_0(\mathfrak{A}_{\mathrm{LG}}(M))$ contains also classes corresponding to products of an equal number of ghost field observables in $H_{-1}(\mathfrak{Obs}(M))$ and antifield observables in $H_1(\mathfrak{Obs}(M))$. $\triangledown$

Conclusions

In this work we studied linearized gravity as a homotopy algebraic quantum field theory on Ricci-flat spacetimes. We built explicitly the $\mathfrak{A}_{\mathrm{LG}}$ functor and we proved that it defines a homotopy AQFT in the locally covariant framework. We tackled the problem of the uniqueness of our quantization prescription, proving only that all restrictions to a fixed Ricci-flat spacetime $\overline{M} \in \mathsf{Loc}_{\mathrm{Ric}}$ are determined uniquely up to weak equivalences.

In particular, we introduced the groupoid for linearized gravity by attaching the arrows given by gauge transformations to the configuration space of gauge fields. This groupoid encodes all information about gauge symmetry of linearized gravity. This information was then condensed in the complex of off-shell gauge fields. We defined a suitable action on this complex and through a critical locus construction we derived the complex of solutions for linearized gravity. The complex of linear observables was later introduced by duality and it was observed that it carries a natural shifted Poisson structure. We proved that this structure is trivial in homology and we trivialized it by two kinds of homotopies which we called retarded/advanced trivializations since they play a role similar to that of retarded/advanced Green operators in ordinary field theory. The construction of these chain homotopies relied on the global hyperbolicity of the background spacetime. Taking the difference between compatible retarded and advanced trivializations allowed us to introduce an unshifted Poisson structure on the complex of observables. This structure is crucial for the canonical quantization of linearized gravity. We proved that the assignment to each Ricci-flat spacetime of quantum observables obtained by implementing canonical commutation relations identifies a homotopical AQFT. In other words, we proved that linearized gravity can be quantized consistently within the homotopical approach.

We also showed that the unshifted Poisson structure is uniquely determined up to homotopies on each fixed Ricci-flat spacetime. This fact allowed us to prove that our quantization prescription identifies a unique up to weak equivalences homotopy AQFT every time that one restricts the attention to a fixed background spacetime. The same uniqueness result remains open when the theory is considered on the entire spacetime category $\mathsf{Loc}_{\mathrm{Ric}}$. This means that we provided a construction of linearized gravity as a homotopy AQFT on the category of Ricci-flat spacetimes, but this construction may still yield homotopy AQFTs that are not weakly equivalent. The (potential) non-uniqueness of the quantum linearized gravity as a homotopy AQFT is linked to properties of the category $\mathsf{Loc}_{\mathrm{Ric}}$ of Ricci-flat spacetimes.

Possible extensions and follow-ups of this thesis are manifold. Firstly, it is possible to consider linearized gravity on non Ricci-flat spacetimes. This corresponds to considering a cosmological constant $\Lambda \neq 0$ in Einstein's equation. Since several solutions of physical interest to Einstein's equation, such as de Sitter spacetime, correspond to non-vanishing Λ, it will be interesting to study how the presence of this additional term affects the constructions we performed. Other extensions may go in the direction of constructing algebraic states for homotopy AQFTs in general, and for linearized gravity in particular. An open point in the homotopy AQFT approach is, indeed, how a suitable notion of algebraic state should be introduced. We recall that an algebraic state is a crucial piece in the description of a physical quantum system in the algebraic formalism as it allows to recover the usual Hilbert space description of quantum theory, see Section 4.1. In the homotopical approach the notion of algebraic state needs to be compatible with the homotopical framework that we have described in the main part of this work. This problem corresponds to the formalization and to the extension of methods, such as Gupta-Bleuler formalism, which are *ad hoc* constructions for electromagnetism. Furthermore, it will be useful to study explicit conditions for the construction of states which fulfill physical consistency requirements. In the standard algebraic formalism this consists in considering Hadamard states, which resemble the singularities of the Minkowski vacuum state in the UV regime and ensure that the quantum fluctuations of all observables are finite. Analogous conditions may be also demanded to states in the homotopical formalism and a thorough analysis of this topic will be crucial in the development of this approach.

Bibliography

[Bae14] C. Baer. “Green-hyperbolic operators on globally hyperbolic spacetimes”. In: *Communications in Mathematical Physics* **333** (2014), pp. 1585–1615. arXiv: 1310.0738 [math-ph].

[BBS19] M. Benini, S. Bruinsma, and A. Schenkel. “Linear Yang-Mills theory as a homotopy AQFT”. In: *Communications in Mathematical Physics* **378** (2019), pp. 185–218. arXiv: 1906.00999 [math-ph].

[BBSS16] C. Becker, M. Benini, A. Schenkel, and R. J. Szabo. “Abelian duality on globally hyperbolic spacetimes”. In: *Communications in Mathematical Physics* **349** (2016), pp. 361–392. arXiv: 1511.00316 [hep-th].

[BD15] M. Benini and C. Dappiaggi. “Models of free quantum field theories on curved backgrounds”. In: *Advances in algebraic quantum field theory*. Ed. by R. Brunetti, C. Dappiaggi, K. Fredenhagen, and J. Yngvason. Mathematical Physics Studies. Springer-Verlag, Heidelberg (2015).

[BDFY15] R. Brunetti, C. Dappiaggi, K. Fredenhagen, and J. Yngvason, eds. *Advances in algebraic quantum field theory*. Mathematical Physics Studies. Springer-Verlag, Heidelberg (2015).

[BDH13] M. Benini, C. Dappiaggi, and T. P. Hack. “Quantum Field Theory on Curved Backgrounds - A Primer”. In: *International Journal of Modern Physics A* **28** (2013), p. 1330023. arXiv: 1306.0527 [gr-qc].

[BDHS14] M. Benini, C. Dappiaggi, T. P. Hack, and A. Schenkel. “A C^*-algebra for quantized principal U(1)-connections on globally hyperbolic Lorentzian manifolds”. In: *Communications in Mathematical Physics* **332** (2014), pp. 477–504. arXiv: 1307.3052 [math-ph].

[BDM14] M. Benini, C. Dappiaggi, and S. Murro. “Radiative observables for linearized gravity on asymptotically flat spacetimes and their boundary induced states”. In: *Journal of Mathematical Physics* **55** (2014), p. 082301. arXiv: 1404.4551 [gr-qc].

[BDS14] M. Benini, C. Dappiaggi, and A. Schenkel. “Quantized Abelian principal connections on Lorentzian manifolds”. In: *Communications in Mathematical Physics* **330** (2014), pp. 123–152. arXiv: 1303.2515 [math-ph].

[Ben16] M. Benini. "Optimal space of linear classical observables for Maxwell k-forms via spacelike and timelike compact de Rham cohomologies". In: *Journal of Mathematical Physics* **57** (2016), p. 053502. arXiv: 1401.7563 [math-ph].

[BFR13] R. Brunetti, K. Fredenhagen, and K. Rejzner. "Quantum gravity from the point of view of locally covariant quantum field theory". In: *Communications in Mathematical Physics* **345** (2013), pp. 741–779. arXiv: 1306.1058 [math-ph].

[BFV03] R. Brunetti, K. Fredenhagen, and R. Verch. "The Generally Covariant Locality Principle – A New Paradigm for Local Quantum Field Theory". In: *Communications in Mathematical Physics* **237** (2003), pp. 31–68. arXiv: 0112041 [math-ph].

[BG11] C. Baer and N. Ginoux. "Classical and Quantum Fields on Lorentzian Manifolds". In: *Springer Proceedings in Mathematics* (2011), pp. 359–400. arXiv: 1104.1158 [math-ph].

[BGP08] C. Baer, N. Ginoux, and F. Pfaeffle. *Wave Equations on Lorentzian Manifolds and Quantization*. (2008). arXiv: 0806.1036 [math.DG].

[BS05] A. N. Bernal and M. Sánchez. "Smoothness of Time Functions and the Metric Splitting of Globally Hyperbolic Spacetimes". In: *Communications in Mathematical Physics* **257** (2005), pp. 43–50. arXiv: gr-qc/0401112 [gr-qc].

[BS19a] M. Benini and A. Schenkel. "Higher Structures in Algebraic Quantum Field Theory". In: *Fortschritte der Physik* **67** (2019), p. 1910015. arXiv: 1903.02878 [hep-th].

[BS19b] S. Bruinsma and A. Schenkel. "Algebraic field theory operads and linear quantization". In: *Letters in Mathematical Physics* **109** (2019), pp. 2531–2570. arXiv: 1809.05319 [math-ph].

[BSS15] M. Benini, A. Schenkel, and R. J. Szabo. "Homotopy colimits and global observables in Abelian gauge theory". In: *Letters in Mathematical Physics* **105** (2015), pp. 1193–1222. arXiv: 1503.08839 [math-ph].

[BSS18] M. Benini, A. Schenkel, and U. Schreiber. "The stack of Yang-Mills fields on Lorentzian manifolds". In: *Communications in Mathematical Physics* **359** (2018), pp. 765–820. arXiv: 1704.01378 [math-ph].

[BSW19] M. Benini, A. Schenkel, and L. Woike. "Homotopy theory of algebraic quantum field theories". In: *Letters in Mathematical Physics* **109** (2019), pp. 1487–1532. arXiv: 1805.087952 [math-ph].

[BSW20] M. Benini, A. Schenkel, and L. Woike. "Operads for algebraic quantum field theory". In: *Communications in Contemporary Mathematics* (2020), p. 2050007. arXiv: 1709.08657 [math-ph].

[DHKS05] W. G. Dwyer, P. S. Hirschhorn, D. M. Kan, and J. H. Smith. *Homotopy Limit Functors on Model Categories and Homotopical Categories*. Mathematical surveys and monographs. American Mathematical Society, Providence (2005).

[Dim92] J. Dimock. "Quantized electromagnetic field on a manifold". In: *Reviews in Mathematical Physics* **4** (1992), pp. 223–233.

[DL12] C. Dappiaggi and B. Lang. "Quantization of Maxwell's equations on curved backgrounds and general local covariance". In: *Letters in Mathematical Physics* **101** (2012), pp. 265–287. arXiv: 1104.1374 [gr-qc].

[Eis97] L. P. Eisenhart. *Riemannian Geometry*. Princeton Landmarks in Mathematics and Physics. Princeton University Press, Princeton (1997).

[FH13] C. J. Fewster and D. S. Hunt. "Quantization of linearized gravity in cosmological vacuum spacetimes". In: *Reviews in Mathematical Physics* **25** (2013), p. 1330003. arXiv: 1203.0261 [math-ph].

[FR12a] K. Fredenhagen and K. Rejzner. "Batalin-Vilkovisky formalism in perturbative algebraic quantum field theory". In: *Communications in Mathematical Physics* **317** (2012), pp. 697–725. arXiv: 1110.5232 [math-ph].

[FR12b] K. Fredenhagen and K. Rejzner. "Batalin-Vilkovisky formalism in the functional approach to classical field theory". In: *Communications in Mathematical Physics* **314** (2012), pp. 93–127. arXiv: 1101.5112 [math-ph].

[FR19] C. J. Fewster and K. Rejzner. *Algebraic Quantum Field Theory - an introduction*. (2019). arXiv: 1904.04051 [hep-th].

[GJ12] P. G. Goerss and J. F. Jardine. *Simplicial homotopy theory*. Progress in Mathematics. Birkhäuser, Basel (2012).

[HE97] S. W. Hawking and G. F. Ellis. *The Large Scale Structure of Space-time*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge (1997).

[Hir09] P. S. Hirschhorn. *Model Categories and Their Localizations*. Mathematical surveys and monographs. American Mathematical Society, Providence (2009).

[HK64] R. Haag and D. Kastler. "An Algebraic Approach to Quantum Field Theory". In: *Journal of Mathematical Physics* **5** (1964), pp. 848–861.

[Hol08] S. Hollands. "Renormalized quantum Yang–Mills fields in curved spacetime". In: *Reviews in Mathematical Physics* **20** (2008), pp. 1033–1172. arXiv: 0705.3340 [gr-qc].

[Hov07] M. Hovey. *Model Categories*. Mathematical surveys and monographs. American Mathematical Society, Providence (2007).

[HW15] S. Hollands and R. M. Wald. "Quantum fields in curved spacetime". In: *Physics Reports* **574** (2015), pp. 1–35. arXiv: 1401.2026 [gr-qc].

[Kha14] I. Khavkine. "Covariant phase space, constraints, gauge and the Peierls formula". In: *International Journal of Modern Physics A* **29** (2014), p. 1430009. arXiv: 1402.1282 [math-ph].

[Kha16] I. Khavkine. "Cohomology with causally restricted supports". In: *Annales Henri Poincaré* **17** (2016), pp. 3577–3603. arXiv: 1404.1932 [math-ph].

[Kha17] I. Khavkine. "The Calabi complex and Killing sheaf cohomology". In: *Journal of Geometry and Physics* **113** (2017), pp. 131–169. arXiv: 1409.7212 [gr-qc].

[KS05] M. Kashiwara and P. Schapira. *Categories and Sheaves*. Grundlehren der mathematischen Wissenschaften. Springer, Berlin (2005).

[Mac98] S. MacLane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics. Springer-Verlag, New York (1998).

[Mor13] V. Moretti. *Spectral Theory and Quantum Mechanics: With an Introduction to the Algebraic Formulation*. UNITEXT. Springer, Milan (2013).

[ONe83] B. O'Neill. *Semi-Riemannian Geometry With Applications to Relativity*. Academic Press, San Diego (1983).

[SDH14] K. Sanders, C. Dappiaggi, and T. P. Hack. "Electromagnetism, local covariance, the Aharonov-Bohm effect and Gauss' law". In: *Communications in Mathematical Physics* **328** (2014), pp. 625–667. arXiv: 1211.6420 [math-ph].

[Wal84] R. M. Wald. *General Relativity*. Chicago University Press, Chicago, USA (1984).

[Wei08] S. Weinberg. *Cosmology*. Oxford University Press, Oxford (2008).

[Wei95] C. A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge (1995).

[Yau18] D. Yau. *Homotopical Quantum Field Theory*. (2018). arXiv: 1802.08101 [math-ph].

Acknowledgments

Ringrazio Claudio e Marco per l'aiuto e il supporto datomi durante la preparazione e la scrittura di questo lavoro. Se sono riuscito a concludere questa fatica, e in un tempo ragionevole, è soprattutto merito loro.

Ringrazio Claudio per avermi fatto scoprire l'esistenza di questo strano ibrido che è la Fisica Matematica. Molte delle mie scelte accademiche degli ultimi anni, a partire dalla tesi triennale, passando per questo lavoro di tesi magistrale, fino a quello che sarà il mio percorso di dottorato nei prossimi tre anni, sono in parte colpa sua.

Ringrazio il Dipartimento di Fisica di Pavia per avermi accolto in questi cinque anni così importanti per la mia formazione.

Ringrazio Diego e Mattia che sono stati i miei veri compagni di avventure in questi ultimi due anni. La laurea magistrale avrebbe avuto sicuramente un sapore diverso senza di loro. Sono convinto che la mia fermezza nell'intraprendere questo percorso sia anche dovuta alla loro compagnia, che mi ha fatto sentire meno solo nelle mie scelte e più compreso. Le tantissime discussioni e i non numerabili sfasi sono stati qualcosa di inestimabile.

A Mattia va anche dato credito per l'aiuto nel portare a termine una dimostrazione in questo lavoro in un momento di panico.

Ringrazio gli amici di questi anni, in particolare Margherita, Riccardo e Samuele (anche Diego e Mattia meriterebbero un posto qui, ma hanno già ricevuto un paragrafo *ad hoc*. So che non saranno esosi). So di essere una persona scostante, quindi grazie.

Infine non posso fare a meno di ringraziare la mia famiglia per avermi sostenuto in tutti questi anni e per aver sempre riposto fiducia in me. Vi ringrazio per avermi sempre permesso di scegliere liberamente, anche se queste scelte mi avrebbero reso più distante. Non finirò mai di ringraziare mia mamma per tutto quello che ha fatto, e continua a fare, per me, nonostante tutte le difficoltà, in questi giorni più che in altri. Grazie!